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Generalizing Van Aubel Using Duality

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A recent paper by DeTemple and Harold [**1**] elegantly utilized the Finsler-Hadwiger theorem to prove Van Aubel's theorem, which states that the line segments, connecting the centers of squares constructed on the opposite sides of a quadrilateral, are congruent and lie on perpendicular lines. This result can easily be generalized by using a less well known "duality" between the concepts *angle* and *side* within Euclidean plane geometry.

Although similar to the general duality between *points* and *lines* in projective geometry, this "duality" is limited. Nevertheless, it occurs quite frequently and examples of these are explored fairly extensively in [**2**]. Obviously this "duality" does not apply to theorems related to or based on the Fifth Postulate (compare [**7**]). For example, the dual to the theorem "three corresponding sides of two triangles equal imply their congruency", namely, "three corresponding angles of two triangles equal imply their congruency", is not valid. (Note, however, that the dual is perfectly true in both non-Euclidean geometries).

The square is self-dual regarding these concepts as it has all angles and all sides congruent. The parallelogram is also self-dual since it has both opposite sides and opposite angles congruent. Similarly, the rectangle and rhombus are each other's duals as shown in the table below:

Rectangle	**Rhombus**
All *angles* congruent	All *sides* congruent
Center equidistant from *vertices*, hence has *circum*circle	Center equidistant from *sides*, hence has *in*circle
Axes of symmetry bisect opposite *sides*	Axes of symmetry bisect opposite *angles*

Furthermore, the *congruent* diagonals of the rectangle has as its dual the *perpendicular* diagonals of the rhombus and is illustrated by the following two elementary results:

(1) The midpoints of the sides of any quadrilateral with congruent diagonals form a rhombus.

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(2) The midpoints of the sides of any quadrilateral with perpendicular diagonals form a rectangle.

The following two dual generalizations of Van Aubel's theorem are proved in [**3**] by generalizing the transformation approach in [**1**]. A vector proof and a slightly different transformation proof for the same generalizations are respectively given in [**2**] and [**4**].

THEOREM 1

If similar rectangles with centers E, F, G and H are erected externally on the sides of quadrilateral ABCD as shown in Figure 1, then the segments EG and FH lie on perpendicular lines. Further, if J, K, L and M are the midpoints of the dashed segments shown, then JL and KM are congruent segments, concurrent with the other two lines.

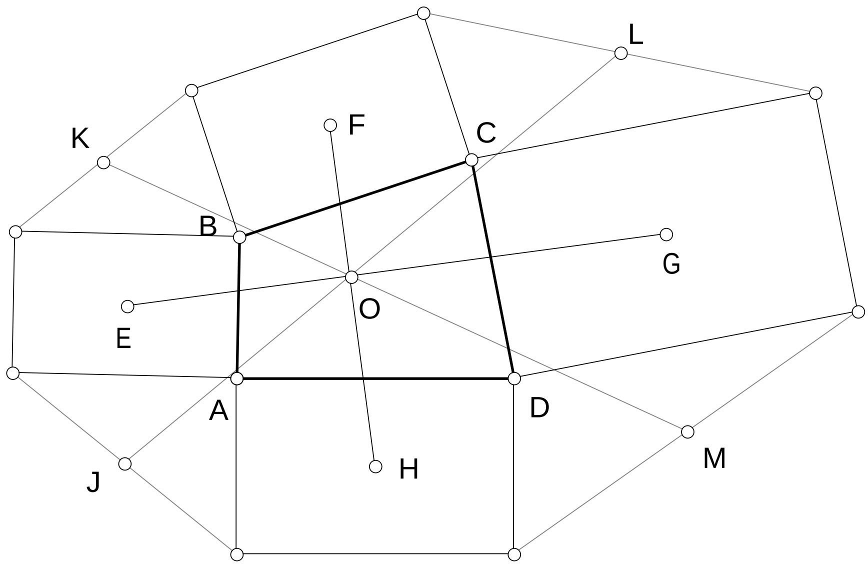

Figure 1

THEOREM 2

If similar rhombi with centers E, F, G and H are erected externally on the sides of quadrilateral ABCD as shown in Figure 2, then the segments EG and FH are congruent. Further, if J, K, L and M are the midpoints of the dashed segments shown, then JL and KM lie on perpendicular lines.

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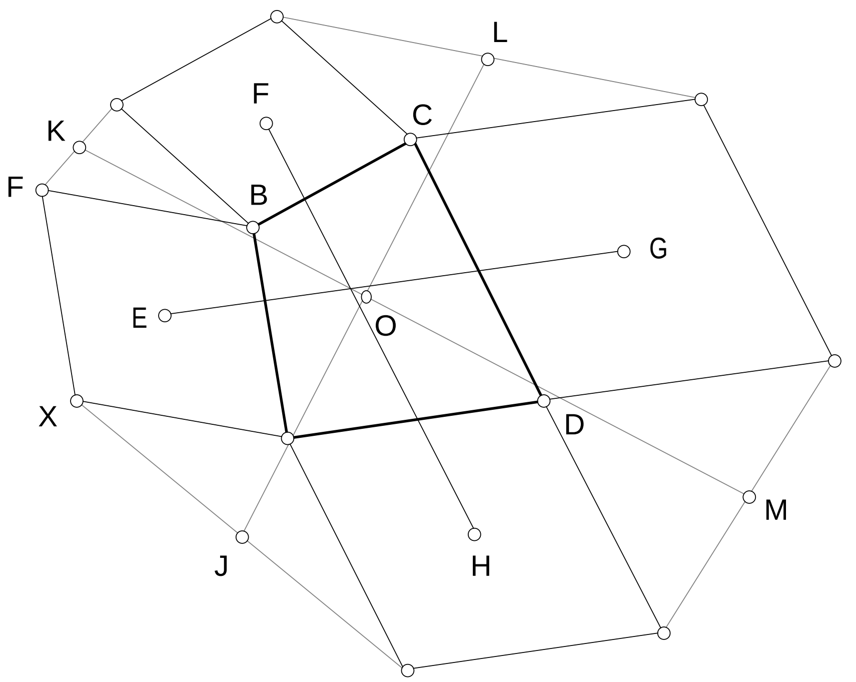

Figure 2

To Theorem 1 the following two properties can be added:

(a) *the ratio of EG and FH equals the ratio of the sides of the rectangles*

(b) *the angle of JL and KM equals the angle of the diagonals of the rectangles*

and to Theorem 2, the following corresponding duals:

(a) *the angle of EG and FH equals the angle of the sides of the rhombi*

(b) *the ratio of JL and KM equals the ratio of the diagonals of the rhombi.*

By combining Theorems 1 and 2, we obtain Van Aubel's theorem, just as the squares are the intersection of the rectangles and rhombi. (For example, for squares it gives us segments *JL* and *KM*, and *EG* and *FH*, respectively congruent and lying on perpendicular lines, as well as concurrent in a single point. In addition, it also follows that all eight angles at the point of intersection are congruent.)

The latter four properties are also contained in the following *self-dual* generalization, which can be proved by using vectors, by complex algebra, or by generalizing the transformation approach used in [**3**]:

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THEOREM 3

If similar parallelograms with centers E, F, G and H are erected externally on the sides of quadrilateral ABCD as shown in Figure 3, then $\frac{FH}{EG} = \frac{XY}{YB}$*, and the angle of EG and FH equals the angle of the sides of the parallelograms. Further, if J, K, L and M are the midpoints of the dashed segments shown, then* $\frac{KM}{JL} = \frac{YA}{XB}$*, and the angle of JL and KM equals the angle of the diagonals of the parallelograms.*

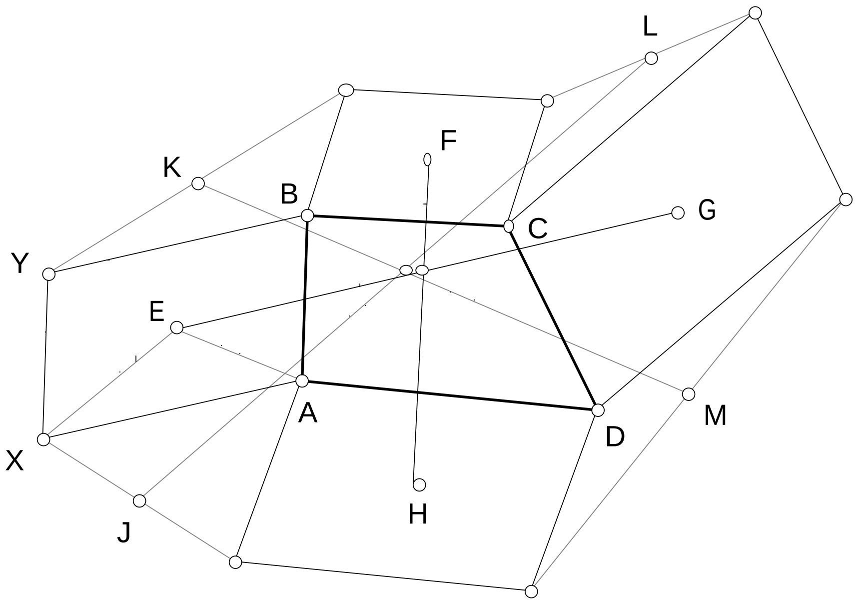

Figure 3

The latter theorem can be further generalized into the following two dual theorems using ideas of Friedrich Bachmann which are extensively developed in [**5**] and [**6**], and which also provide a powerful technique and notation, giving automatic proofs for problems of this kind.

THEOREM 4

If similar parallelograms are erected externally on the sides of quadrilateral ABCD and similar triangles

$$XP_0A,\ AP'_0B,\ QP_1B,\ BP'_1C,\ RP_2C,\ CP'_2D,\ SP_3D,\ DP'_3A$$

are constructed as shown in Figure 4, and E, F, G and H are the respective midpoints of the segments $P_iP'_{i+1}$ *for* $i = 0, 1, 2, 3$*, then* $\frac{FH}{EG} = \frac{XY}{YB}$*, and the angle of EG and FH equals the angle of the sides of the parallelograms.*

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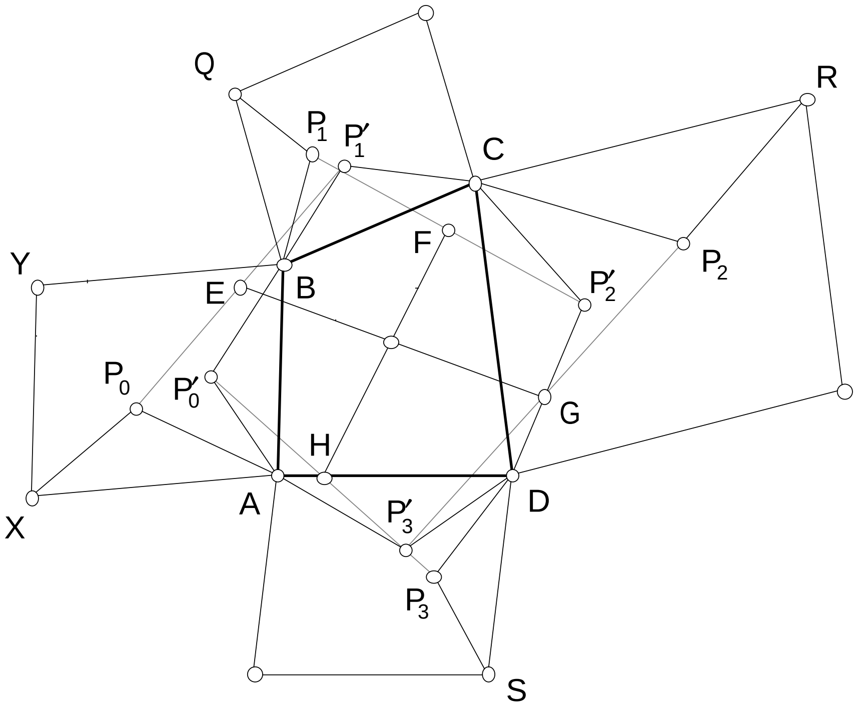

Figure 4

THEOREM 5

If similar parallelograms with centers E, F, G, and H are erected externally on the sides of quadrilateral ABCD and I_i *are the midpoints of the dashed segments as shown in Figure 5, parallelograms are constructed with* I_i *as centers as well as similar triangles* TP_0E, EP'_0F, QP_1F, FP'_1G, RP_2G, GP'_2H, SP_3H, HP'_3E*, and K, L, M and J are the respective midpoints of the segments* $P_iP'_{i+1}$ *for i = 0, 1, 2, 3, then* $\frac{KM}{JL} = \frac{YA}{XB}$*, and the angle of JL and KM equals the angle of the diagonals of the parallelograms.*

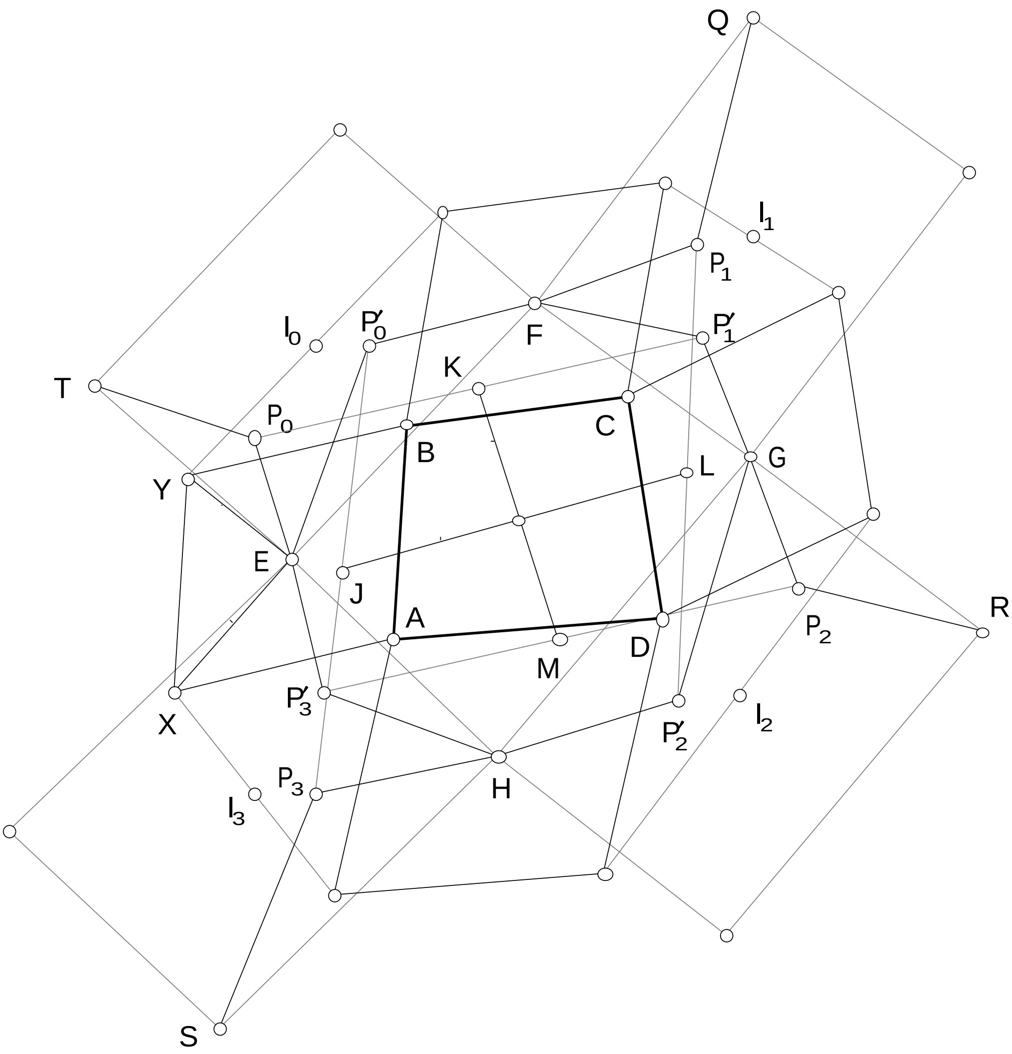

Figure 5

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