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Dual Generalizations of Van Aubel's theorem

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In Euclidean plane geometry there exists an interesting, although limited, duality between the concepts *angle* and *side*, similar to the general duality between *points* and *lines* in projective geometry. Perhaps surprisingly, this duality occurs quite frequently and is explored fairly extensively in [2].

The square is self-dual regarding these concepts as it has all angles, as well as all sides equal. Similarly, the rectangle and rhombus are each other's duals as shown in the table below:

Rectangle	**Rhombus**
All *angles* equal	All *sides* equal
Center equidistant from *vertices*, hence has *circum* circle	Center equidistant from *sides*, hence has *in* circle
Axes of symmetry bisect opposite *sides*	Axes of symmetry bisect opposite *angles*

It also appears that the *equal* diagonals of the rectangle has as its dual the *perpendicular* diagonals of the rhombus. An example is given by the following two elementary results (the proofs of which are left to the reader):

(1) The midpoints of the sides of any quadrilateral with equal diagonals form a rhombus (see Figure1).

(2) The midpoints of the sides of any quadrilateral with perpendicular diagonals form a rectangle (see Figure 2).

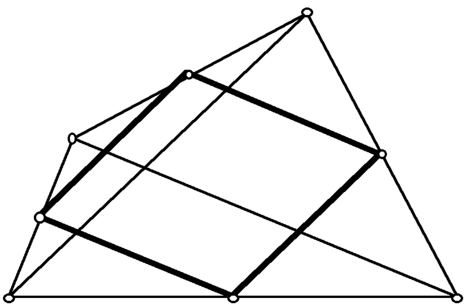

Figure 1

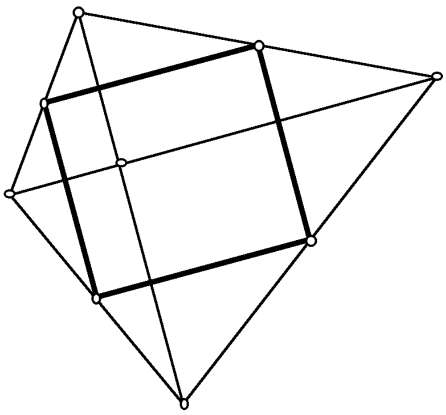

Figure 2

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This paper will now provide intriguing extensions of some results in a paper by DeTemple & Harold [1], by utilizing the above-mentioned duality between the rectangles and rhombi. The following dual Theorems 1 and 2 are generalizations of their Problem 7 involving squares.

Theorem 1

(a) If similar rectangles ABCD and AB'C'D' share a vertex at A (where both rectangles are labelled clockwise, then the segments BB' and DD' lie on perpendicular lines (see Figure 3).

(b) Further, if P is the point at which BB' and DD' intersect, then the line CC' also passes through P and the line AP is perpendicular to it.

(c) Also, if O and O' are the respective centers of rectangles ABCD and AB'C'D', then $OO' = \frac{1}{2} CC'$.

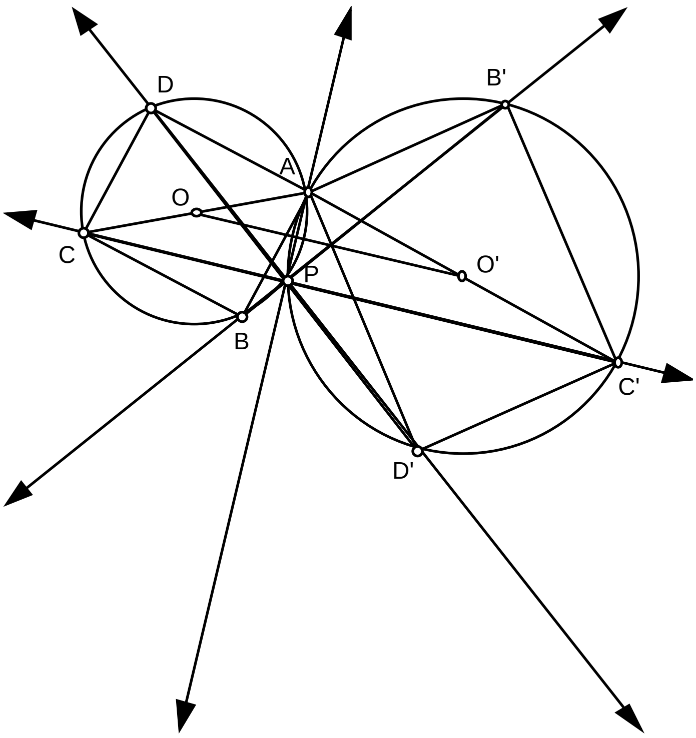

Figure 3

Proof

(a) The spiral similarity (k; 90°) about point A maps ABB' onto ADD' (k = AD/AB = AD'/AB' from the similarity of the rectangles). Therefore BB' and DD' are contained in lines that cross at 90°.

(b) Draw the circumscribing circles of each of these rectangles. Since ∠B'PD' = 90° as shown in (a), it follows that these circles intersect at A and P (∠B'PD' = 90° =

∠B'AD' on diameter BD'). Since AC' is a diameter it follows that ∠APC' = 90°. Similarly, ∠APC = 90° and therefore CPC' is a straight line.

(c) In ACC', the points O and O' are the midpoints of sides AC and AC', and therefore OO' //= $\frac{1}{2}$ CC'.

Theorem 2

(a) If similar rhombi ABCD and AB'C'D' share a vertex at A (where both rhombi are labelled clockwise), then the segments BB' and DD' are congruent (see Figure 4).

(b) If O and O' are the respective centers of rhombi ABCD and AB'C'D', then OO' = $\frac{1}{2}$ CC'.

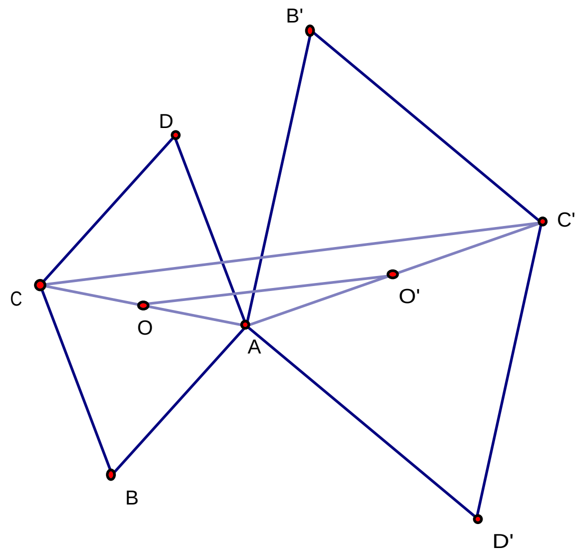

Figure 4

Proof

(a) A rotation of size ∠BAB' around point A maps ABB' onto ADD', showing that the triangles are congruent, and therefore BB' = DD'.

(b) In ACC', the points O and O' are the midpoints of sides AC and AC', and therefore OO' //= $\frac{1}{2}$ CC'. Also note that unlike the preceding result, CC' does not necessarily pass through the intersection point of the lines through BB' and DD'.

The following two dual theorems are generalizations of Problem 8 in [1] involving squares.

Theorem 3

If two similar rectangles ABCD and AB'C'D' share a vertex, then the midpoints, Q and S, of the segments B'D and BD' together with centers R and T form another similar rectangle TSRQ (see Figure 5).

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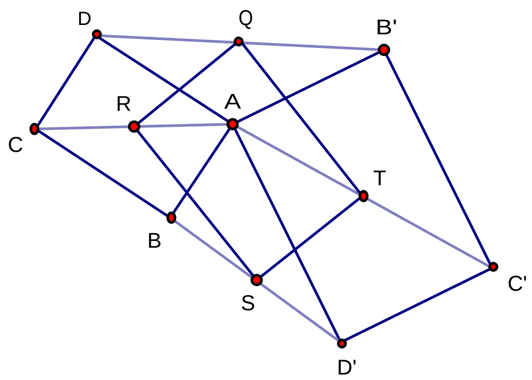

Figure 5

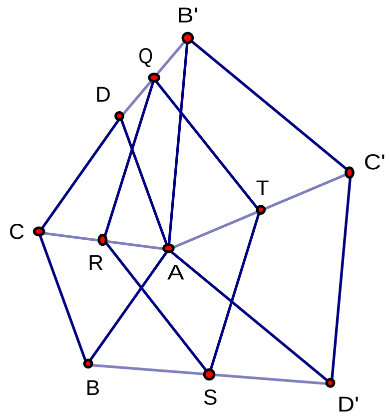

Figure 6

Theorem 4

If two similar rhombi ABCD and AB'C'D' share a vertex, then the midpoints, Q and S, of the segments B'D and BD' together with centers R and T form another similar rhombus TSRQ (see Figure 6).

Proof

Both Theorems 3 and 4 depend on the following useful general result given and proved in [2]: "Let F_0 and F_1 denote two directly similar figures in the plane, where $P_1 \in F_1$ corresponds to $P_0 \in F_0$ under the given similarity. Let $r \in (0,1)$, and define $F_r = \{(1-r)P_0 + rP_1\}$. Then F_r is also directly similar to F_0."

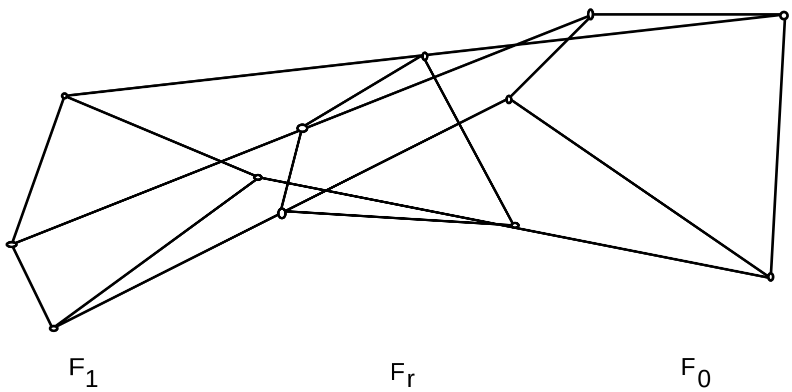

Figure 7

An example of the theorem is illustrated in Figure 7 where the figures are quadrilaterals and $r = \frac{1}{2}$. In the case of Theorems 3 and 4, we also have $r = \frac{1}{2}$, but F_0 and F_1 share a common vertex.

The following two dual Theorems 5 and 6 are generalizations of Van Aubel's theorem presented as Problem 15 in [1].

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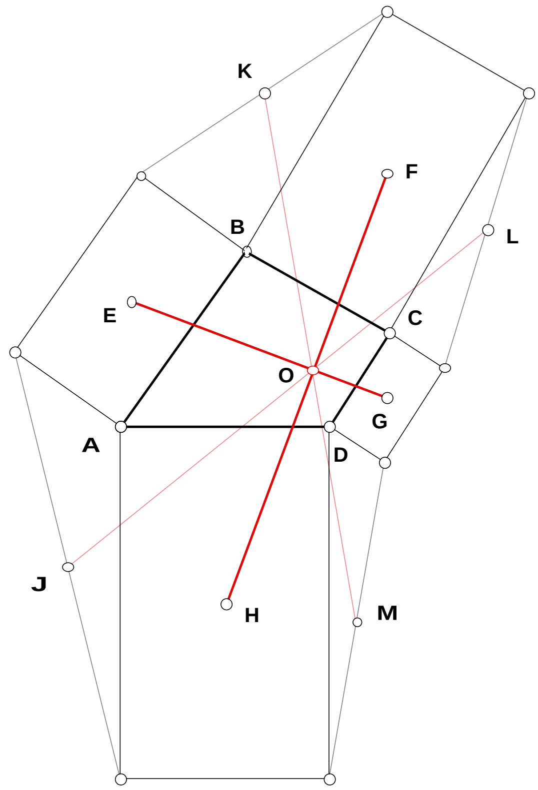

Figure 8

Theorem 5

If similar rectangles with centers E, F, G and H are erected externally on the sides of quadrilateral ABCD as shown in Figure 8, then the segments EG and FH lie on perpendicular lines. Further, if J, K, L and M are the midpoints of the dashed segments shown, then JL and KM are congruent segments, concurrent with the other two lines.

Proof

The configurations in Theorems 1 and 3 provide the keys to a proof. By Theorem 3, the similar rectangles with diagonals EF and GH have a common vertex at the midpoint of AC (see Figure 9). Similarly, the similar rectangles with diagonals FG and EH have a common vertex at the midpoint of BD. Theorem 1 shows that EG and FH lie on perpendicular lines and that KM and JL are concurrent with EG and FH.

By erecting similar rhombi as shown by the dotted lines on the sides of quadrilateral EFGH as shown, we obtain the same configuration as in Theorem 6, from which follows that KM and JL are congruent.

From Theorem 1, also note that EG is twice the breadth of similar rectangle PQRS, and FH is twice its length. Similarly, the common length of KM and JL is twice the length of a diagonal of rectangle PQRS.

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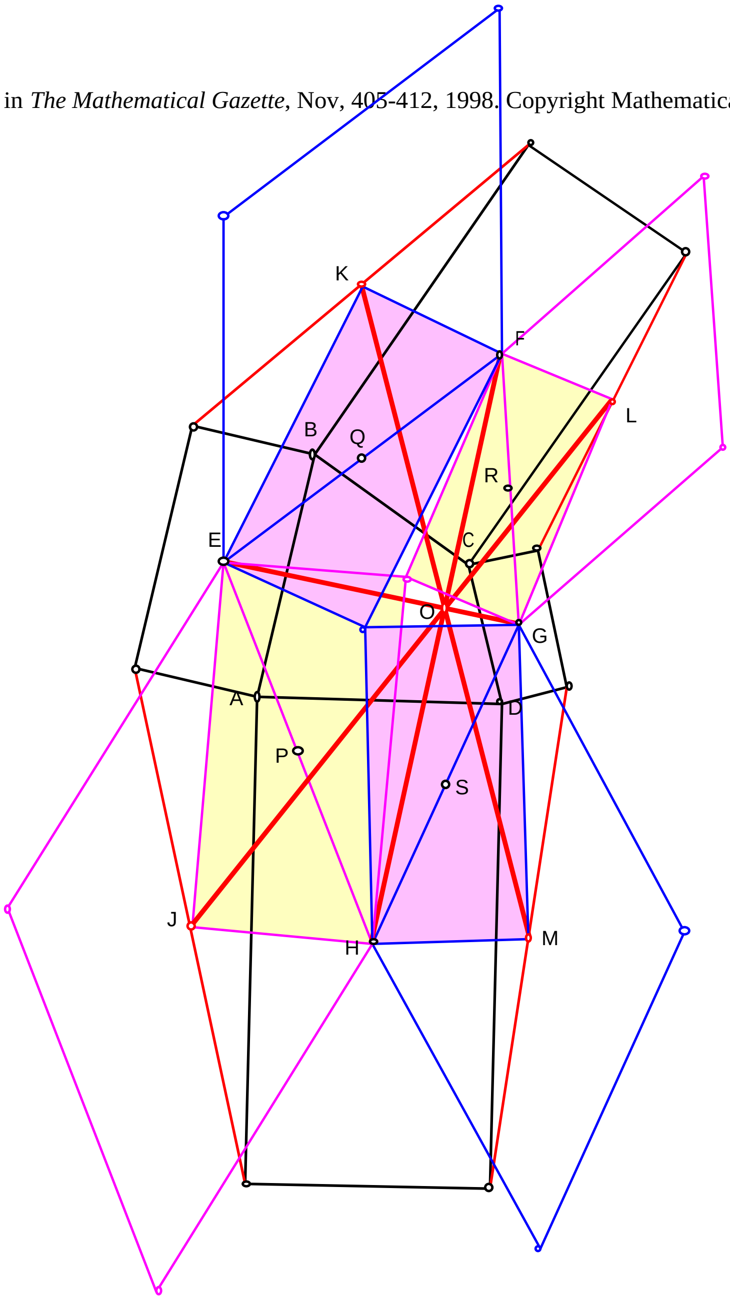

Figure 9

Another interesting observation is that FH and EG are angle bisectors of the angles formed by the other two lines. By drawing the circumcircles of rectangles with diagonals EF and FG, it follows that O lies on both circles. Therefore, ∠FOL = ∠FGL (on chord FL) and ∠FOK = ∠FEK (on chord FK). But since FGL is similar to FEK, it follows that ∠FOL = ∠FOK. Thus, FH is the angle bisector of ∠KOL, and similarly EG is the angle bisector of ∠KOJ.

Theorem 6

If similar rhombi with centers E, F, G and H are erected externally on the sides of quadrilateral ABCD as shown in Figure 10, then the segments EG and FH are congruent. Further, if J, K, L and M are the midpoints of the dashed segments shown, then JL and KM lie on perpendicular lines.

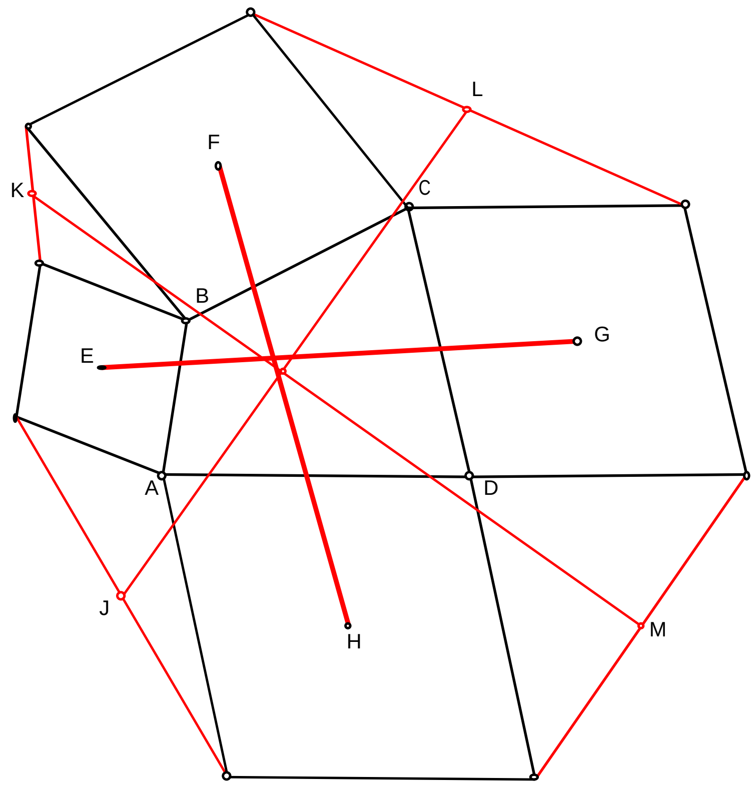

Figure 10

Proof

The configurations in Theorems 2 and 4 provide the keys to a proof. By Theorem 4, the similar rhombi with diagonals EF and GH have a common vertex at the midpoint of AC (see Figure 11). Similarly, the similar rhombi with diagonals FG and EH have a common vertex at the midpoint of BD. Theorem 2 shows that EG and FH are congruent.

By erecting similar rectangles as shown by the dotted lines on the sides of quadrilateral EFGH as shown, we obtain the same configuration as in Theorem 5, from which follows that KM and JL lie on perpendicular lines.

From Theorem 2, also note that the common length of EG and FH is twice the length of a side of similar rhombus PQRS. Similarly, KM is twice the length of diagonal QS and JL is twice the length of diagonal PR.

Remark

Different transformation and vector proofs for the respective orthogonality and congruency of segments EG and FH in Theorems 5 and 6 are given in [2] and [3]. The extension of these two results to include segments JL and KM using the approach of [1], seems to be new.

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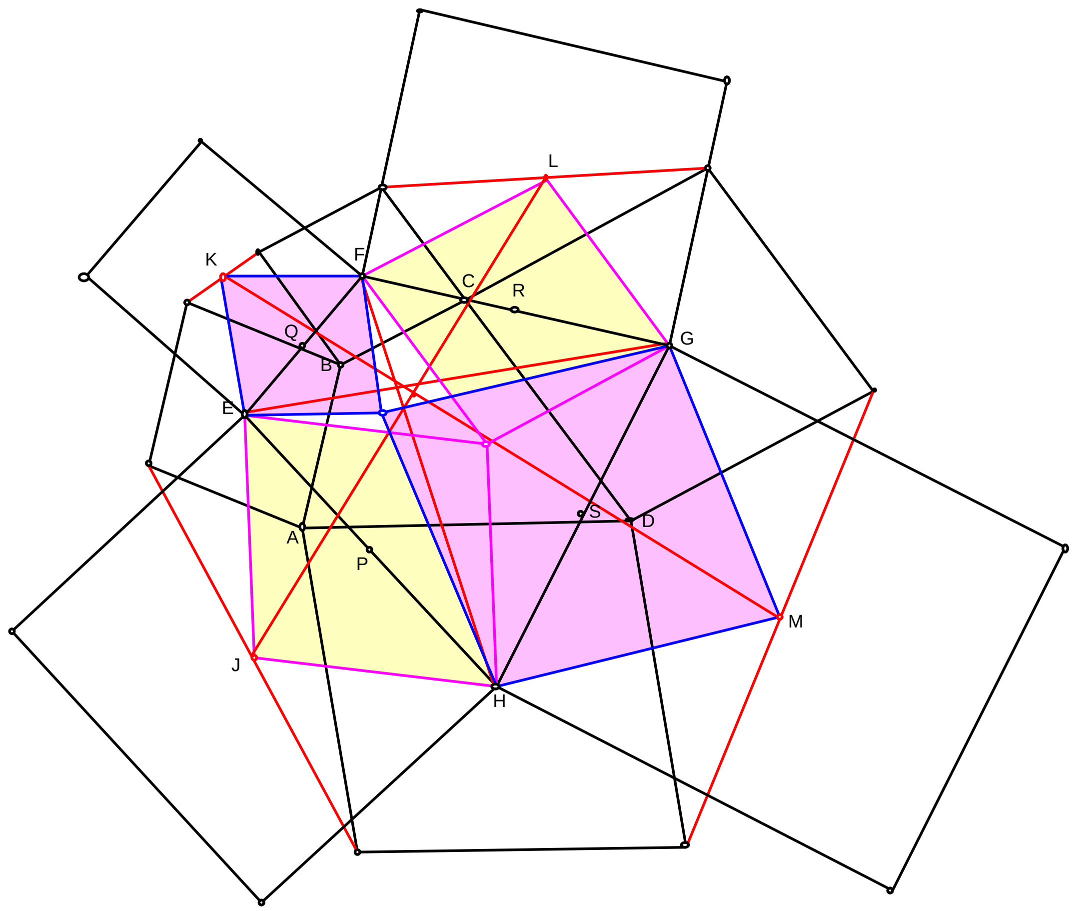

Figure 11

These dual generalizations are particularly appealing, since the special cases with squares are easily obtained from them. For example, just as the squares are the intersection of the rectangles and rhombi, we obtain Van Aubel's theorem by combining Theorems 5 and 6. (I.e. for squares it gives us segments JL and KM, and EG and FH, respectively congruent and lying on perpendicular lines, as well as concurrent in a single point. In addition, it also follows that all eight angles at the point of intersection are equal.)

Note: Zipped Sketchpad 3 sketches related to this article and some further generalizations of Van Aubel can be downloaded from http://mzone.mweb.co.za/residents/profmd/aubel.zip

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