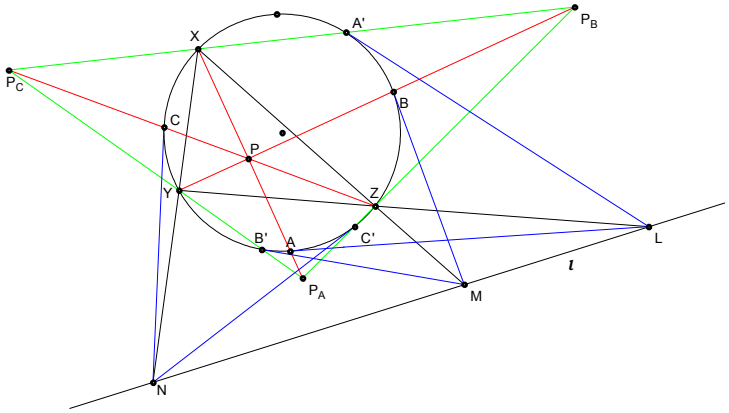
Let X, Y, Z be three points on a conic C, and l a line not passing through X, Y or Z. Let $L \equiv YZ.l$; $M \equiv ZX.l$; $N \equiv XY.l$. The tangents from L to C touch C at A and A', those from M touch C at B, B'; from N touch at C, C'. Then these contact points can be labelled so that XA, YB, ZC are concurrent, as are the triples XA, YB', ZC'; XA', YB, ZC'; and XA', YB', ZC.



The proof uses general homogeneous coordinates: a point P is represented by the ordered triple (x, y, z). We do not specify what x represents: what determines the point is any pair of independent ratios obtained from the three coordinates. Consequently, for all $k \neq 0$, P is (kx, ky, kz), and we can write symbolically that $P \equiv kP$. We can also assign the coordinates of any four independent points or lines.

We take XYZ as the triangle of reference, with $X(1,0,0)$, $Y(0,1,0)$, $Z(0,0,1)$. We also take l as the unit line $x + y + z = 0$. As these are not areal coordinates, l is not necessarily the line at infinity, though we may choose (later) to interpret it in this way.

Since L is on YZ it has the form $\mu Y + \nu Z$, so is $(0, \mu, \nu)$. But it is on l, so $\mu + \nu = 0$. Thus $L = (0, -1, 1)$. Cyclically $M = (1, 0, -1)$, $N = (-1, 1, 0)$.

The conic C, passing through X, Y, Z, may be taken as $f^2yz + g^2zx + h^2xy = 0$. (We shall see the reason for taking the coefficients as squares in a moment.) Every point on C can be expressed in terms of a parameter λ in the form $(f^2\lambda(\lambda+1), g^2(\lambda+1), -h^2\lambda)$. The tangent at (α, β, γ) is $(g^2\gamma + h^2\beta)x + (h^2\alpha + f^2\gamma)y + (f^2\beta + g^2\alpha)z = 0$. At a point with parameter λ this is $g^2h^2x + h^2f^2\lambda^2y + f^2g^2(\lambda+1)^2z = 0$, which is equivalent to $\dfrac{1}{f^2}x + \dfrac{\lambda^2}{g^2}y + \dfrac{(\lambda+1)^2}{h^2}z = 0.$

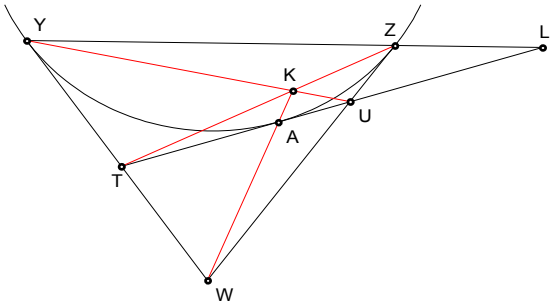
Now suppose a tangent passes through $L(0, -1, 1)$: we find that $-h^2\lambda^2 + g^2(\lambda+1)^2 = 0$. Thus $\big(g(\lambda+1) + h\lambda\big)\big(g(\lambda+1) - h\lambda\big) = 0$. This gives the values $\dfrac{-g}{g+h}$ and $\dfrac{-g}{g-h}$ for λ. Substituting these values we find that the contact points A and A' are $\left(-\dfrac{f^2}{g+h}, g, h\right)$ and $\left(\dfrac{f^2}{g-h}, -g, h\right)$. We shall take the first of these as A. We find that B is $\left(f, -\dfrac{g^2}{h+f}, h\right)$, B' is $\left(f, \dfrac{g^2}{h-f}, -h\right)$, C is $\left(f, g, -\dfrac{h^2}{f+g}\right)$, C' is $\left(-f, g, \dfrac{h^2}{f-g}\right)$.

Now all points on XA can be expressed as $\mu X + A$, so have the form $(*, g, h)$, where $*$ is a quantity that we do not need explicitly, and can take any value. Thus XA passes

through $P(f, g, h)$. By symmetry, so do YB and ZC. Hence these three lines are concurrent.

Points on YB' are $(f, *, -h)$, i.e. $(-f, *, h)$, and those on ZC' are $(-f, g, *)$. Thus these lines meet at P_A $(-f, g, h)$, which also lies on XA. Hence XA, YB', ZC' meet at P_A. The other triples follow in the same way, giving four intersection points $P(f, g, h)$, $P_A(-f, g, h)$, $P_B(f, -g, h)$, $P_C(f, g, -h)$.

If C is an ellipse, then X and A are on opposite sides of YZ, etc. The tangent at A, whose parameter is $-g/(g + h)$, is $(g+h)^2x + f^2y + f^2z = 0$; the tangent at B $(-h/(h + f))$ is $g^2x + (h+f)^2y + g^2z = 0$; and at C is $h^2x + h^2y + (f+g)^2z = 0$. Also the tangents at X (∞), Y (0) and $Z(-1)$ are $h^2y + g^2z = 0$, $h^2x + f^2z = 0$, $g^2x + f^2y = 0$. The six tangents form a circumscribed hexagon $QRSTUV$, where $X \in QR$, $Y \in RS$, etc. Solving the equations we find that $Q = (f^2 + 2hf, -g^2, h^2)$, with S and U following by cyclic interchange; and $R = (f^2 + 2fg, g^2, -h^2)$, T and V being obtained cyclically. We now find that $Q + T = (2hf, 2gh, 2h^2)$, which reduces to $P(f, g, h)$, thus QT passes through P. Similarly so do RU and SV.



Each side of the degenerate hexagon $WZUATY$ touches the conic, so Brianchon's theorem applies. Thus WA, ZT, UY meet in a point K, say. In triangle WUT apply Ceva's theorem to K and Menelaus's theorem to the line LZY. We find that $UA/AT = -UL/LT$. Hence A, L are harmonic conjugates with respect to U, T. In particular, if l is the line at infinity, so $L \to \infty$ and $UL/LT \to -1$, then $A \to$ midpoint of UT. Similarly, B and C become the midpoints of VQ and RS. This proves your conjecture.

We can deduce some more results. We find that $R + Q = (*, 0, 0) = X$; and $R - Q = \left(f(g-h), g^2, -h^2\right) = L'$, say, being the harmonic conjugate of X with respect to Q, R. Similarly $M' = \left(-f^2, g(h-f), h^2\right)$, $N' = \left(f^2, -g^2, h(f-g)\right)$. These are collinear, lying on $\dfrac{-f+g+h}{f}x + \dfrac{f-g+h}{g}y + \dfrac{f+g-h}{h}z = 0$. Let us call this l'.

Also $L' + ghL = \left(f(g-h), g(g-h), h(g-h)\right) = P(f, g, h)$. So LL' and, by symmetry, MM', NN' also pass through P.

The intersections $(QR.TU)$, $(RS.UV)$, $(ST.VQ)$ lie on a line p whose equation is $\dfrac{g+h}{f}x + \dfrac{h+f}{g}y + \dfrac{f+g}{h}z = 0$. Thus $QRSTUV$ is inscribed in a conic. Since $p = l + l'$, these three lines meet in a point, which is $\left(f(g-h), g(h-f), h(f-g)\right)$. There are many other pairs or triples of lines that meet on p.

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