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A Dual to a BMO Problem[1]

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In the 1999 Round 2 of the British Mathematical Olympiad (BMO) the following interesting problem was posed (see [4] for a proof]:

"Let *ABCDEF* be a hexagon which circumscribes a circle *S*. The circle *S* touches *AB*, *CD*, *EF* at their midpoints *P*, *Q*, *R* respectively. Let *X*, *Y*, *Z* be the points of contact of *S* with *BC*, *DE*, *FA* respectively. Prove that *PY*, *QZ*, *RX* are concurrent."

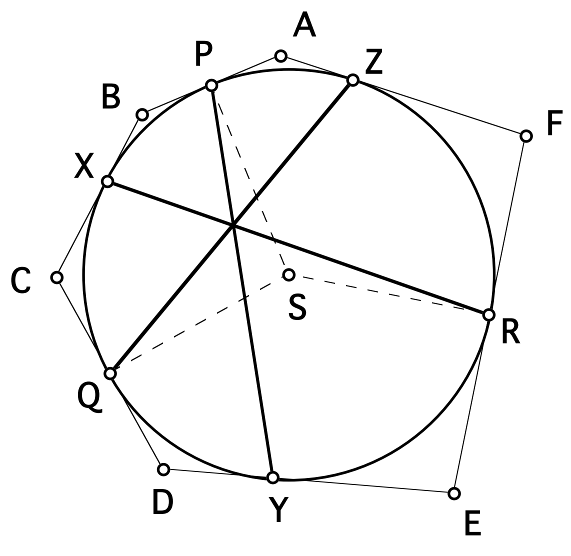

Figure 1

This result can be reformulated differently in the following equivalent form: "If the *perpendicular* bisectors of the alternate *sides* of a hexagon *circumscribed* around a circle are concurrent at the center of the circle, then the lines connecting opposite *tangential points* are concurrent (see Figure 1)." From a *side-angle* duality in plane geometry which is explored fairly extensively in [1], as well as used in [2] and [3] to

[1] An accompanying *Sketchpad* sketch of this article illustrating the results can be directly downloaded in zipped form from http://mysite.mweb.co.za/residents/profmd/bmo.zip

discover new generalizations of Van Aubel's theorem, I then immediately conjectured the follow dual:

"If the *angle* bisectors of the alternate *angles* of a hexagon *inscribed* in a circle are concurrent at the center of the circle, then the lines connecting opposite *vertices* are concurrent."

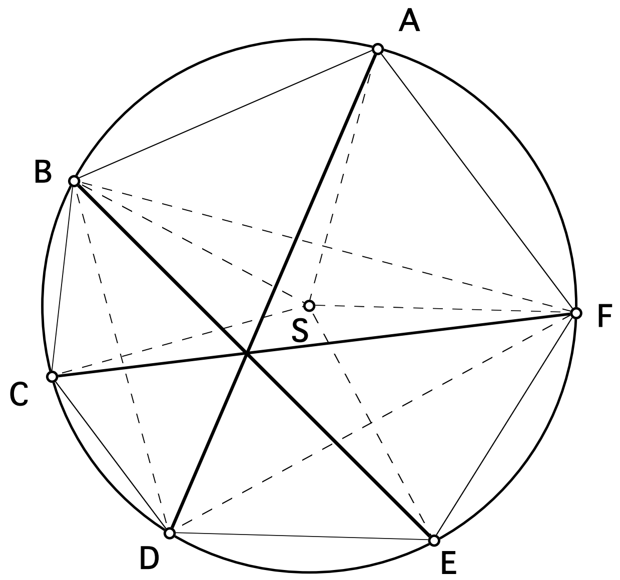

Figure 2

This dual can very easily be proved as follows (in a manner similar to that of the original result). Suppose hexagon *ABCDEF* is given with the angle bisectors of alternate angles *A*, *C* and *E* concurrent at *S* (see Figure 2). Triangles *ABS* and *AFS* are congruent (Angle *SAF* = angle *SFA* = angle *SAB* = angle *SBA* since *AS* is an angle bisector and *SA* = *SF* = *SB*). Therefore, *AB* = *AF* which implies that the angles inscribed on these two chords are also equal; i.e. angle *ADB* = angle *ADF*. Thus, *DA* is an angle bisector of angle *BDF* in triangle *BDF*. In the same way can be shown that *BE* and *FC* are angle bisectors of the other two angles in triangle *BDF*, and completes the proof since the angle bisectors of any triangle are concurrent.

The duality between the two results extends somewhat further. From the proof above, it follows that the *cyclic* hexagon has three pairs of adjacent *sides* equal (adjacent to angles *A*, *C*, and *E*). Similarly, it is easy to prove in the first result, that the *circum* hexagon has three pairs of adjacent *angles* equal (adjacent to sides *AB*, *CD*, and *EF*). Although the concurrency result unfortunately does not generalize, the latter observation easily generalizes to the following two duals:

"If the *perpendicular* bisectors of the alternate *sides* of a *circum* 2*n*-gon ($n > 1$) are concurrent at the center of the *incircle*, then it has *n* distinct pairs of equal adjacent *angles*."

"If the *angle* bisectors of the alternate *angles* of a *cyclic* 2*n*-gon ($n > 1$) are concurrent at the center of the *circumcircle*, then it has *n* distinct pairs of equal adjacent *sides*."

Interestingly, the first generalization above in the case of a quadrilateral, becomes an isosceles trapezium, and is therefore both circumscribed and cyclic (called an *isosceles circum trapezium* in [1]). Similarly, the dual generalization becomes a kite, and is therefore both cyclic and circumscribed (called a *right kite* in [1]). Note also that the former has a line of symmetry through a pair of opposite sides, whereas the latter has a line of symmetry through a pair of opposite angles.

Although this side-angle duality is limited, it is nevertheless a useful conceptual tool for constructing new conjectures, and hopefully some new results.

REFERENCES

1. M. de Villiers, *Some Adventures in Euclidean Geometry*. University of Durban-Westville: Durban, South Africa, 1996.
   (A slightly revised version of this book is now available as PDF download or in book form at: http://www.lulu.com/product/paperback/some-adventures-in-euclidean-geometry/5414956)
2. M. de Villiers, Dual generalizations of Van Aubel's theorem, *The Mathematical Gazette*, (November 1998), 405-412.
3. M. de Villiers, More on dual Van Aubel generalizations, *The Mathematical Gazette*, (March 2000), 121-122.
4. *The UK Mathematics Trust Yearbook*, 1998-1999, 92-94.