Bradley's Theorem, an Analogue and their Generalisation

Michael de Villiers

RUMEUS, University of Stellenbosch

profmd1@mweb.co.za

INTRODUCTION

Generalisation is one of the most important processes in mathematics, and features whenever we make a leap from observing a particular pattern while exploring specific cases to formulating a general conjecture covering all cases. While this sort of inductive generalisation is probably the most common form of generalisation in mathematics, there is also another type which I would like to call a 'deductive' or 'reflective' generalisation. This usually happens when we generalise from a particular infinite class of cases to a larger infinite class which includes the former. So for example, we can extend and generalise the arithmetic of positive whole numbers to include negative numbers, and from there go further to include the rational and irrational numbers, then to all the real numbers, and eventually the complex numbers. This sort of generalisation is no longer based on inductive generalisation, but on maintaining the same structural integrity.

In contrast, in geometry, we tend to remain fixated on only studying triangles and quadrilaterals without exploring the possible further generalisation of observations to other polygons. This unfortunate practice often leads to learners and students having very limited intuitions and understanding of geometry and proof. For example, I have regularly found my student teachers, knowing that opposite sides parallel for a quadrilateral ensures that the quadrilateral then also has opposite sides equal, often over-generalise when they are asked about a hexagon with opposite sides parallel, with them erroneously believing that in such a hexagon the opposite sides will necessarily also be equal.

The example presented in this paper provides an interesting, accessible geometry problem posed by Bradley (2004) that generalises to higher polygons. This 'deductive' generalisation is made possible by the proof which immediately shows that the same argument applies to other polygons, and provides yet another example of what has been called the 'discovery' function of proof (De Villiers, 1990).

BRADLEY'S THEOREM

The following result is from Bradley (2004) where it was stated without proof, and was left to the reader to prove: If $ABCD$ is a tangential quadrilateral as shown in Figure 1 with its sides AB, BC, CD and DA respectively touching its incircle at K, L, M and N, then the respective incentres P, Q, R and S of triangles AKN, BLK, CML and DNM lie on its incircle (and obviously form a cyclic quadrilateral). In addition, $\angle PQR = (\angle NKL + \angle KLM)/2$ etc.

A dynamic geometry sketch to explore the theorem is provided for the reader at:
http://dynamicmathematicslearning.com/concyclic-incentres-bradley.html

Before continuing further, readers are challenged to prove Bradley's theorem for themselves.

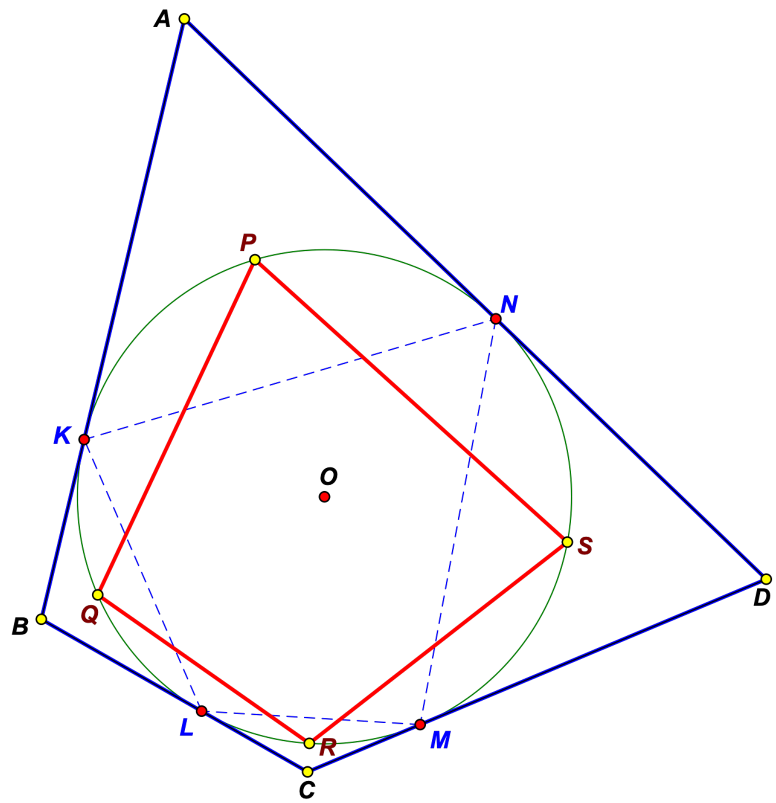

FIGURE 1

PROOF

We will first prove that the respective incentres P, Q, R and S of triangles AKN, BLK, CML and DNM lie on the incircle of $ABCD$. Consider Figure 2 which shows incentre Q, deliberately drawn not to lie on the incircle, so as not to inadvertently assume what we have to prove.

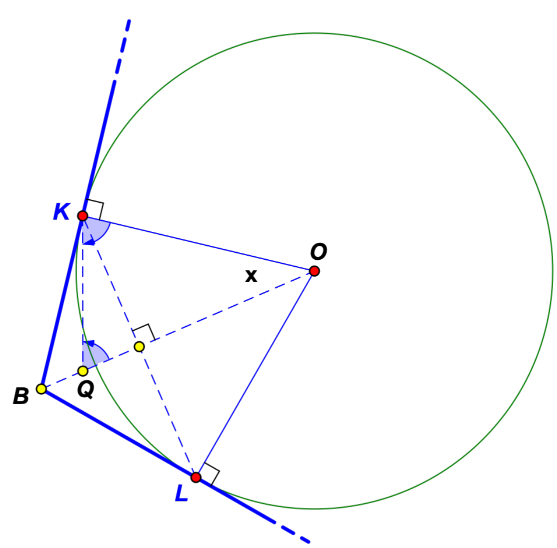

FIGURE 2

Clearly $KBLO$ is a kite since BK and BL are equal tangents and OK and OL are equal radii. Since Q is the incentre of ΔBLK it must lie on the angle bisector of $\angle KBL$ which is the axis of symmetry BO of the kite. Let $\angle KOB = x$. Then $\angle KBO = 90° - x$ since KB is a tangent at K to the circle. Since the diagonals of a kite are perpendicular to one another, it follows that $\angle BKL = 180° - (90° + 90° - x) = x$. Since KQ is the angle bisector of $\angle BKL$, it next follows that $\angle BKQ = \frac{1}{2}x$ and therefore $\angle OKQ = \angle OKB - \angle BKQ = 90° - \frac{1}{2}x$. But $\angle OQK = 90° - x + \frac{1}{2}x = 90° - \frac{1}{2}x$ (exterior angle of ΔKBQ). Hence, $\angle OKQ = \angle OQK$, which implies that $OK = OQ$, and therefore the point Q must lie on the incircle. Similarly, using exactly the same argument, we can show that the other incentres P, R and S also lie on the incircle.

Let us now prove that $\angle PQR = (\angle NKL + \angle KLM)/2$. Consider Figure 3. If we let $\angle QPR = x$ and $\angle QRP = y$, then $\angle QOR = 2x$ and $\angle QOP = 2y$ (angle at centre is twice the angle on circumference). Since $KUOT$ has a pair of opposite right angles at U and T, it is cyclic. Hence, $\angle NKL = 180° - 2y$. Similarly, we can show that $\angle KLM = 180° - 2x$. Therefore $(\angle NKL + \angle KLM)/2 = 180° - x - y$. But $\angle PQR = 180° - x - y$ (angle sum in triangle ΔPQR), which completes the proof. We can use the same argument to similarly show that $\angle QRS = (\angle KLM + \angle LMN)/2$, etc.

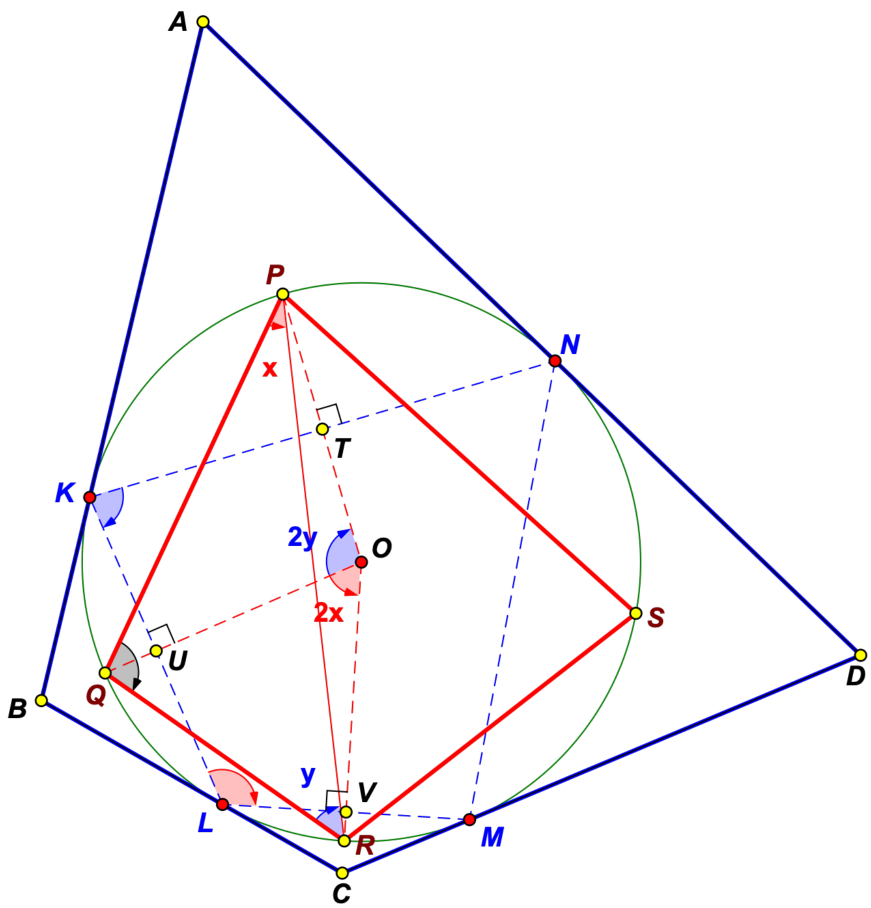

FIGURE 3

AN ANALOGOUS RESULT WITH CIRCUMCENTRES

Interestingly, we can formulate an analogous result with the circumcentres of isosceles triangles formed by a cyclic quadrilateral as follows: If $ABCD$ is a cyclic quadrilateral as shown in Figure 4, then the respective circumcentres P, Q, R and S of triangles AOB, BOC, COD and DOA form a tangential quadrilateral. In addition, $\angle SPQ = \angle DAB + \angle ABC - (\angle PSR + \angle PQR)/2$, etc.

A dynamic version illustrating the result is also available at the URL given earlier.

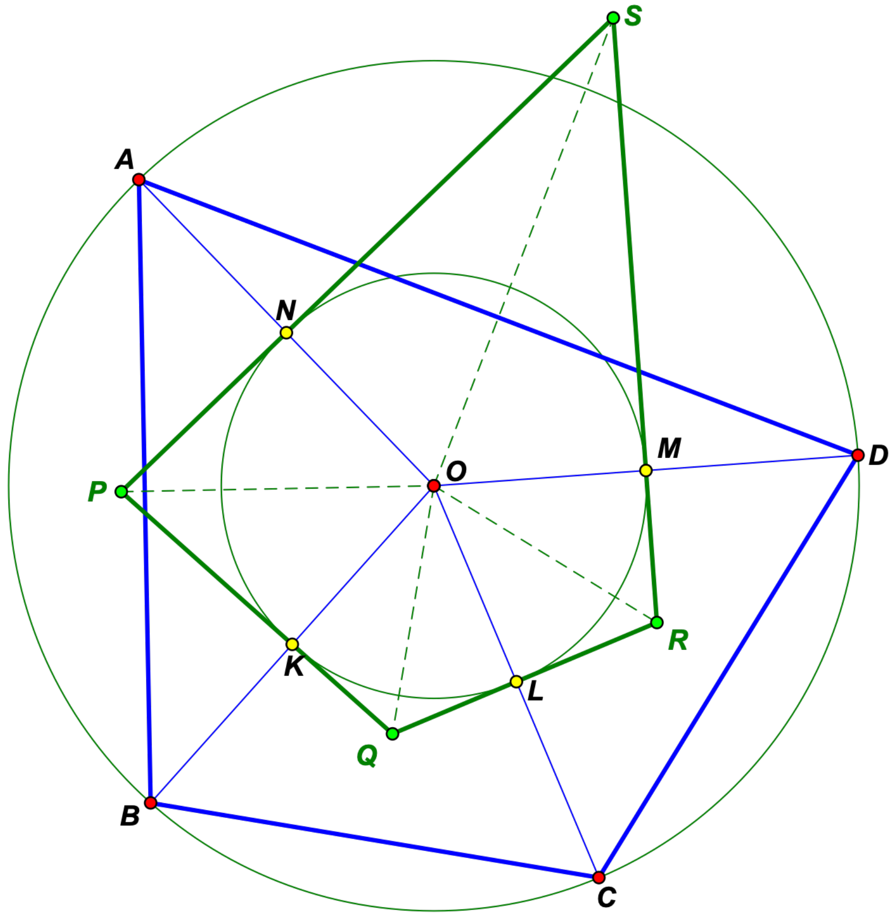

FIGURE 4

PROOF

We shall first prove that $ABCD$ is a tangential quadrilateral. We will do this by proving that its angle bisectors are concurrent at O (see De Villiers, 2020). Since P is the circumcentre of ΔAOB, we have $PA = PO$. Similarly, $SA = SO$. Hence, $APOS$ is a kite and $\angle PNO = 90°$. In the same way, it follows that $PBQO$ is a kite and $\angle PKO = 90°$. Further note that $APBO$ is also a kite, hence $\angle NOP = \angle KOP$. Therefore, $\Delta PNO \equiv \Delta PKO$ (AAS), and hence PO is the angle bisector of $\angle NPK$. Similarly, we can show that QO, RO and SO are the respective angle bisectors of $\angle PQR$, $\angle QRS$ and $\angle RSP$, and this completes the proof that $ABCD$ is a tangential quadrilateral.

Next we prove $\angle SPQ = \angle DAB + \angle ABC - (\angle PSR + \angle PQR)/2$. Consider Figure 5. Since $AYOX$ has a pair of opposite right angles it is a cyclic quadrilateral. We thus have $\angle POS = 180° - \angle DAB$. If we let $\angle PSO = x$, then from the sum of the angles in triangle POS it follows that $\angle SPO = \angle DAB - x$. Similarly, $\angle POQ = 180° - \angle ABC,$ and if we let $\angle PQO = y$ then $\angle OPQ = \angle ABC - y$. Therefore, $\angle SPQ = \angle DAB + \angle ABC - (x + y)$. But $(x + y) = (\angle PSR + \angle PQR)/2$, and therefore completes the proof.

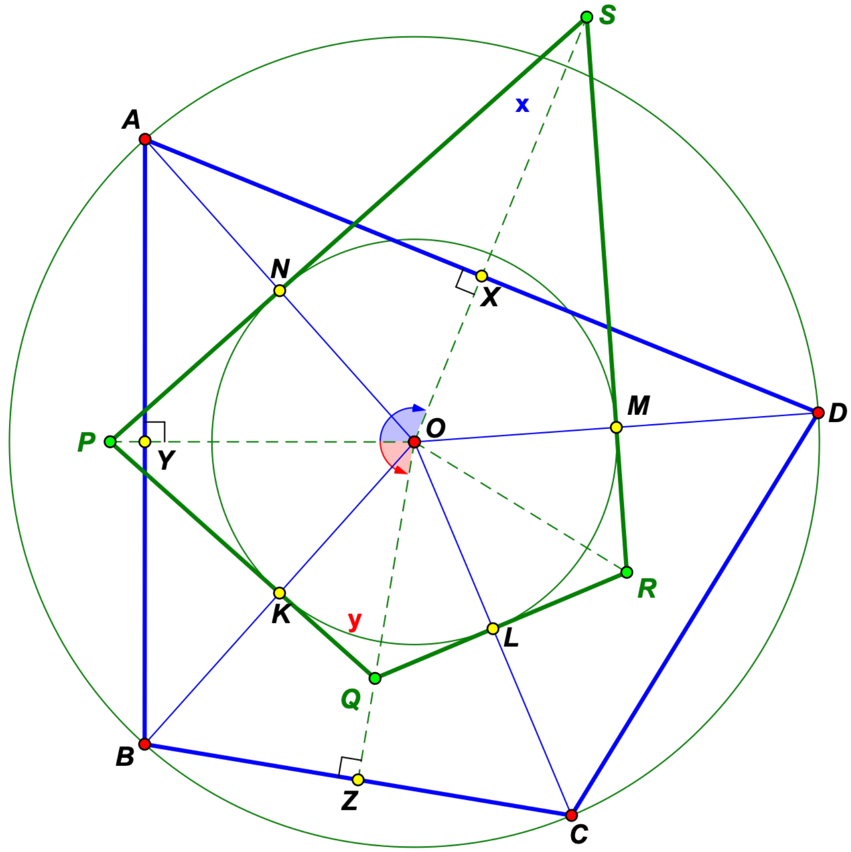

FIGURE 5

LOOKING BACK & GENERALISING

Polya (1945) mentions in his 4th and final step of problem solving that much can be gained by looking back and carefully reflecting on one's proofs. Looking back carefully at the proofs of both Bradley's theorem and its analogue, it should be easy to see that the proofs do not depend on the number of vertices of the starting polygon. Hence, they respectively generalise to tangential polygons and cyclic polygons since exactly the same proofs would apply.

Dynamic sketches for a tangential pentagon as well as for a cyclic pentagon are available at the URL given at the start, but both cases are illustrated in Figure 6.

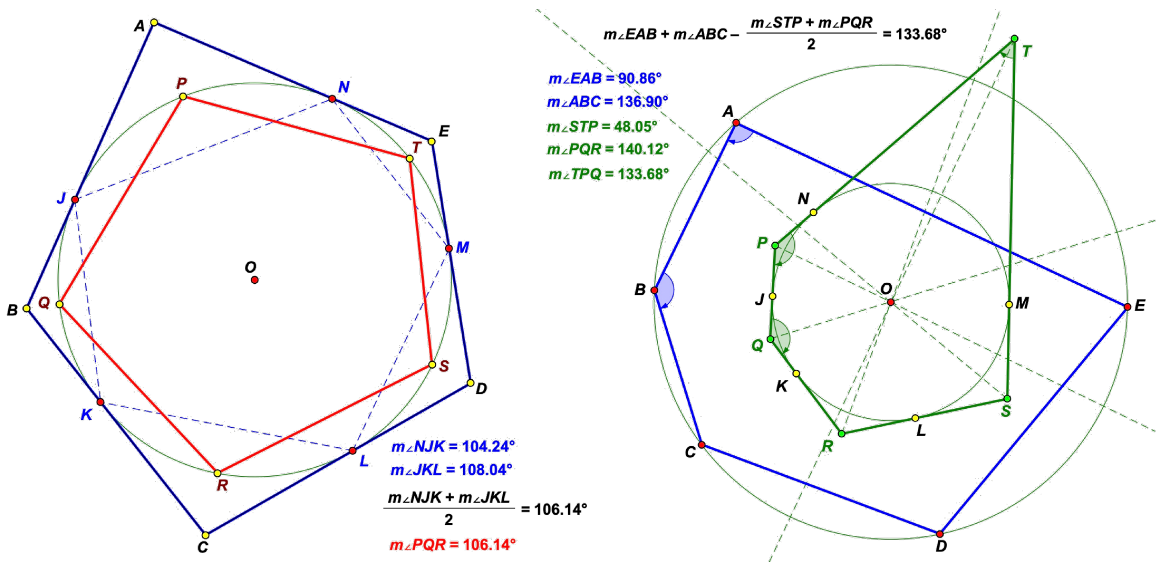

FIGURE 6

Note that both results as stated above, and their generalisations, only remain valid if both $ABCD$ and $PQRS$ remain convex. It is possible to extend both results further to include concave and crossed cases, but this becomes tricky, and requires careful use of directed angles, directed triangles, etc.

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