Excerpt from: De Villiers, M. & Jahnke, H.N. (2024). "The Intimate Interplay between Experimentation and Deduction: Some Classroom Implications", pp. 2551-2586. In Sriraman, B. (Ed.). Handbook of the History and Philosophy of Mathematical Practice, Springer Nature, Switzerland.

counterexamples by the instructor or students can also deepen understanding, help combat resistant misconceptions, and potentially enhance generic critical thinking.

In the next section, we will briefly examine the use of scientific principles and results within mathematics. In particular, we will focus on the value and use of balancing points (centroids) in the discovery and explaining of some geometric theorems.

6 Using Scientific Principles in Mathematics

The implementation of the principles of physics has since ancient times been a productive approach in many areas of mathematics for not only discovering new results, but also logically explaining (proving) them. Introducing principles of physics into a mathematical theory amounts to adding new axioms/hypotheses to the set of axioms so far accepted. Of course, when a new result is found and proved by such additional principles it still remains an important question to investigate whether it can also be proved in the established axiomatic domain of mathematics proper.

In the following, we restrict ourselves to the application of principles of statics to geometry since these seem to provide the only examples accessible to school teaching.

One of the most famous mathematicians and scientists of antiquity was undoubtedly Archimedes (c.287 – c.212 BC) of Syracuse.

Archimedes used the centroids (centres of gravity) of figures, and the law of the lever, to deduce the volumes of spheres, cones, and pyramids (Heath 1897; Polya 1954; Hawking 2006). Another important figure was the Italian engineer Giovanni Ceva (1647–1734 AD) who in 1678 published a book "De lineis rectis se invicem secantibus statica constructio" exposing a comprehensive approach to elementary geometry by means of static considerations.

The law of the lever can be stated as follows: "Magnitudes are in equilibrium at distances reciprocally proportional to their weights" and is illustrated in Fig. 8. Algebraically, when in balance (equilibrium), it can be formulated as $W_1 \times L_1 = W_2 \times L_2$, where W_i and L_i are, respectively, the weights and distances to the fulcrum. In practice, it is a familiar occurrence to children playing on a seesaw where, for example, a heavy adult would need to be balanced by two or more smaller kids on the other side.

Over the years, the authors have used the experimental and theoretical exploration of balancing points (centroids/centres of gravity) of triangles, quadrilaterals, and other polygons with undergraduate and postgraduate students as well as in

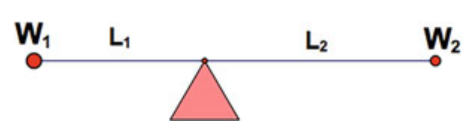


Fig. 8 Law of the balanced lever

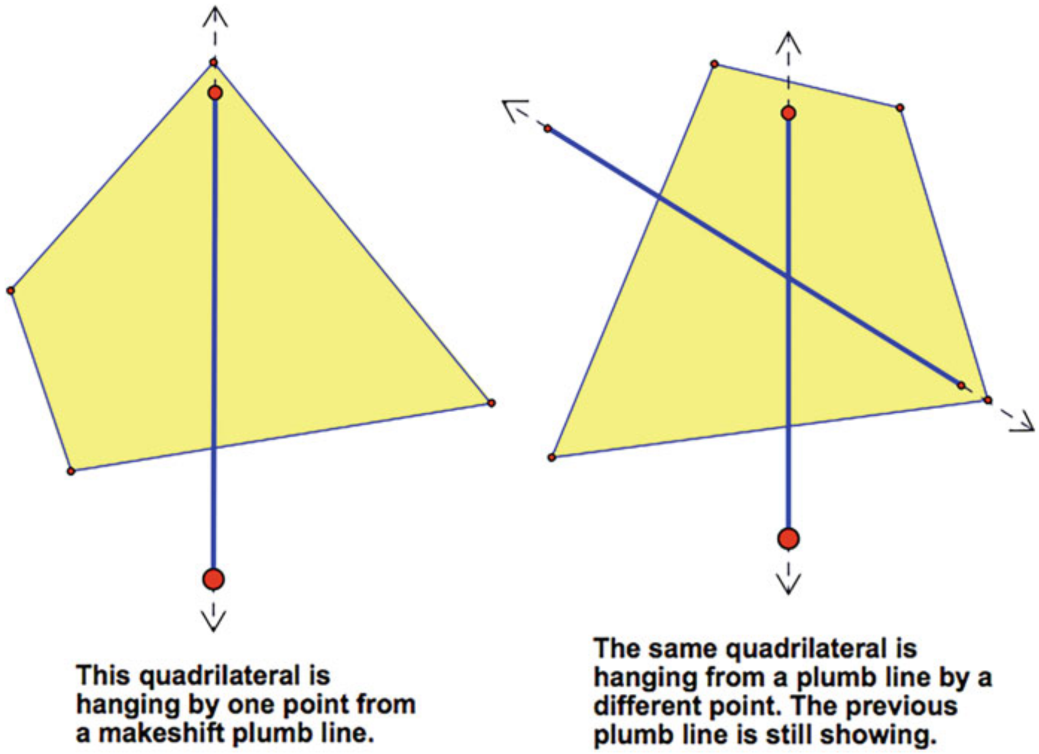


Fig. 9 Finding a centre of gravity with a plumb line

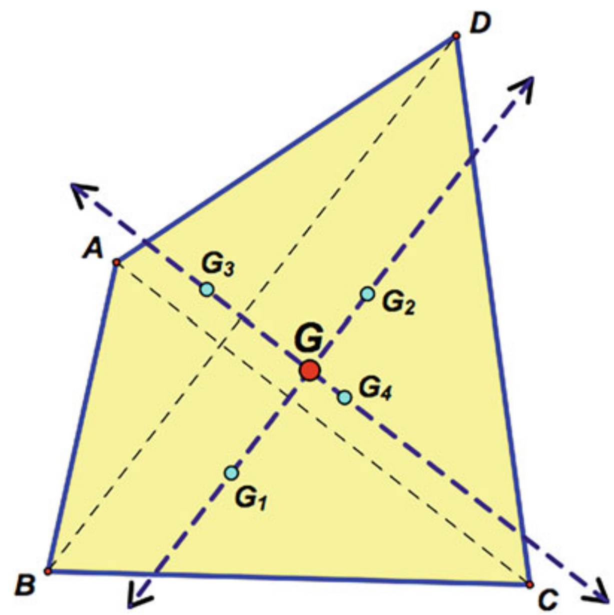
workshops with in-service teachers. The balancing point of a two- or three-dimensional object is called its centre of gravity. In architecture and engineering, accurately locating balancing points is extremely important for designing stable structures that do not collapse (Gordon 1978). The centre of gravity of a two-dimensional object can easily be found through experimentation. For example, in order to locate the approximate centre of gravity of the polygon, students can be asked to balance a cardboard polygon on the tip of a pencil or an eraser.

Another way of experimentally locating the centre of gravity of a cardboard polygon is shown in Fig. 9. The polygon is hanging on a string that is attached near its edge. The string is acting as a carpenter's plumb line, which provides the carpenter with a line perpendicular to the ground. The balancing point or centre of gravity is then located where these two lines cross. When doing experiments like these to locate centres of gravity, the results are obviously subject to some experimental error.

After dealing with the following theoretical method of finding the centroid of a cardboard (lamina[4]) quadrilateral *ABCD* (see Fig. 10), students can be encouraged to carry out the geometric construction on their physical models, and to compare the experimental finding with the theoretical location:

[4] In physics, a lamina is a two-dimensional object with uniform density and negligible thickness.

Fig. 10 Finding the centroid of a cardboard quadrilateral



1) Find G_1 and G_2, the respective centroids of triangles *ABC* and *ACD*, then the balancing point of the whole quadrilateral must lie somewhere on the line connecting these two centroids.
2) Find G_3 and G_4, the respective centroids of triangles *ABD* and *BCD*, then the balancing point of the whole quadrilateral must lie somewhere on the line connecting these two centroids.
3) Hence, the balancing point of *ABCD* must lie at the intersection of the two lines in 1) and 2).

Generally, the results of comparing the experimentally found balancing point and the theoretical location agree fairly well, unless of course major experimental or construction errors were made. Of some pedagogical interest too is to have students drag a dynamic quadrilateral with its constructed lamina centroid so that it becomes concave. Much to their surprise, and some cases even with some mild bewilderment, students will find that the balancing point could actually move outside of the quadrilateral as shown in Fig. 11. In order to balance a concave cardboard quadrilateral like that, one would have to attach a thin (comparatively weightless) wire to the centroid in order to balance it. While the balancing point moving outside of a figure is not something uncommon – for example, the rim of a wheel also has its centroid "outside" the rim, which is why it needs to be connected with spokes to its balancing point, i.e., the centre of the circle forming the rim – students nonetheless find this quite a surprising revelation.

From a sporting perspective this observation also provides a scientific explanation why the so-called Fosbury flop is so much more efficient for high jumping than the old-fashioned straddle technique. By bending their bodies in an arc as shown in Fig. 12, high jumpers are able to move their centres of gravity slightly outside their

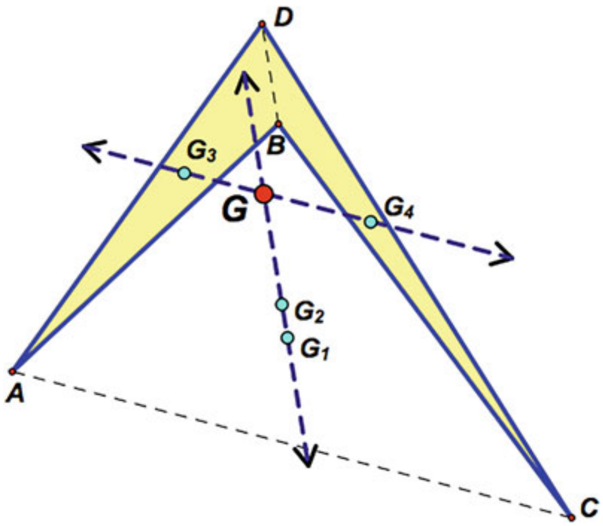


Fig. 11 Balancing point outside concave quadrilateral

Fig. 12 Fosbury flop in high jumping

body as they go over the cross bar, hence saving a few centimeters, which could be the difference between winning or not.

One can also demonstrate the theoretical solution for finding the balancing point of a cardboard quadrilateral quite nicely in another way with dynamic geometry. Since the relative weights of the cardboard triangles *ABD* and *BCD* are determined by their areas, we can determine the relative weights, respectively, concentrated at the centroids G_3 and G_4 by measuring the areas of triangles *ABD* and *BCD* as shown in Fig. 13. Now using the lever law of Archimedes, we can easily determine the balancing point (centroid) of the weights at G_3 and G_4. In particular, dilating G_3 from G_4 as centre and a scale factor of $\frac{areaABD}{(areaABD+areaBCD)}$ gives us the required balancing point G_3'. And doing this construction on the same sketch as the earlier one in Fig. 10 will show students that it is the same balancing point G as before.

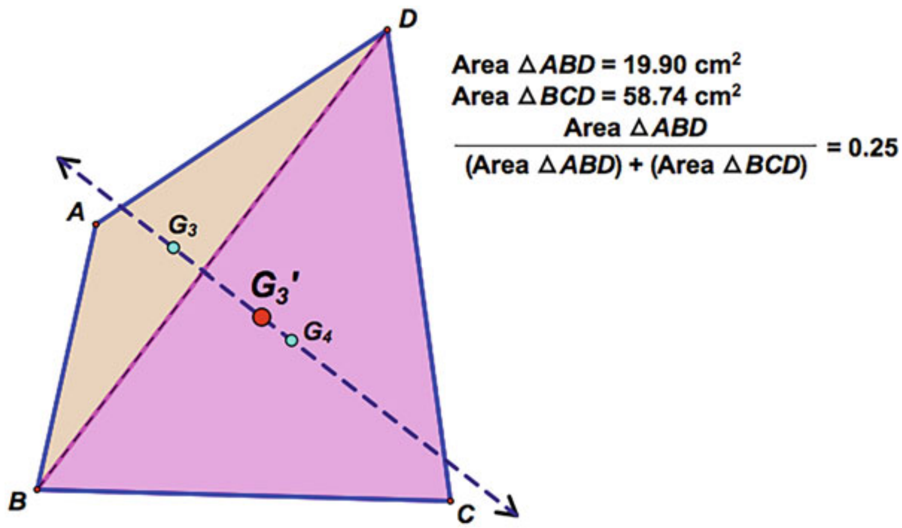


Fig. 13 Using areas as measure of cardboard weight

Saving the theoretical construction of the balancing point of a cardboard quadrilateral centroid as a tool in dynamic geometry also enables us to easily find the balancing point of a cardboard pentagon by dividing it in two different ways into a quadrilateral and a triangle, drawing the two lines between the respective centroids and finding their intersection. Alternatively, one can again use the law of the lever on a division of a pentagon into a quadrilateral and a triangle and determining their respective centroids and areas. This approach is easily extended to hexagons, etc.

A different physical way of analyzing balancing points (centroids) of polygons is to imagine equal point masses located at the vertices, and connected with thin wire of negligible weight to make it a stable structure. Where would such a structure balance? In the case of the triangle, the point mass balancing point is the same as that of the cardboard triangle, lying at the point of concurrency of the medians, but except for a parallelogram, it is not generally the case for a quadrilateral that the point mass centroid coincides with its lamina (cardboard) centroid.

As demonstrated by the French mathematician Pierre Varignon (1654–1722), the point mass centroid of a quadrilateral with equal weights at the vertices is located at the centre of the Varignon parallelogram formed by the midpoints of the sides (e.g., Hanna and Jahnke 2002). As shown in the first figure in Fig. 14, the point mass centroid does not coincide with the lamina centroid. The second figure in Fig. 14 also shows another interesting geometric property that is easy to explain (prove) using arguments from physics. For example, given equal point masses at the vertices of *ABCD* it follows that the two weights at *A* and *C* would balance at the midpoint *K* of the diagonal *AC*. Since the same applies to the two weights at *B* and *D*, balancing at the midpoint *L* of the diagonal *BD*, it follows that altogether as a system, the four weights would balance at the midpoint of *KL*, which implies that it coincides with the point mass centroid (the Varignon centre of the Varigon parallelogram).

Similarly, by considering different weights at the vertices of a triangle, just like Giovanni Ceva did (see above), we can also deduce, from the principles of mechanics, the celebrated concurrency theorem of Ceva (compare Hanna and Jahnke 2002).

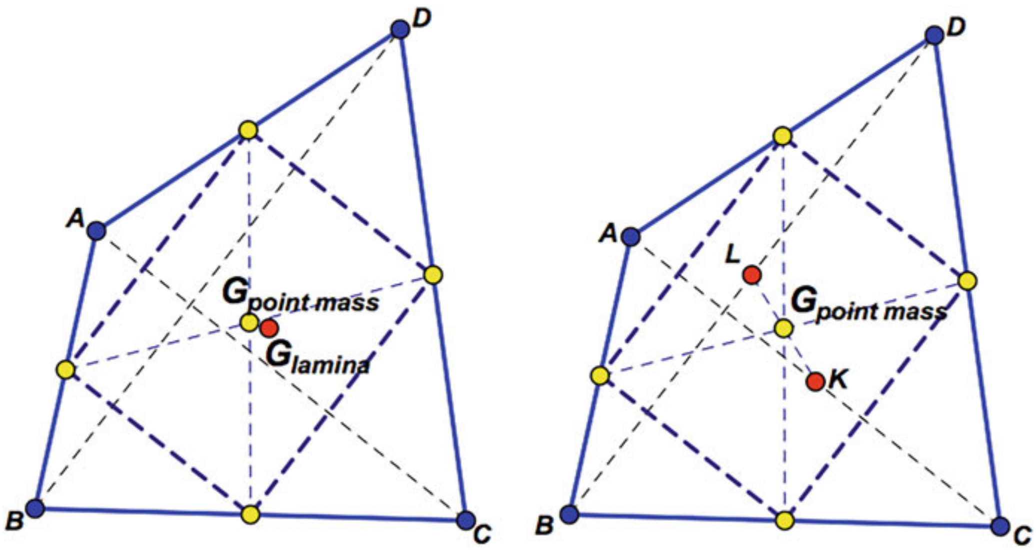


Fig. 14 Point mass centroid and lamina centroid

Fig. 15 Mechanical demonstration of Fermat point

Lastly, it is not hard to design a dynamic geometry exploration for students, and a guided worksheet to prove from coordinate geometry, that the coordinates of the centroid of a triangle can be found from the average of the coordinates of its vertices. This can be further extended by using a weighted average to find the coordinates for the Ceva point for different point mass weights located at the vertices, and easily extended further to higher polygons.

For a short outlook on other ways of using means from physics to understand mathematical relations, we hint at applying physical models and phenomena to illustrate mathematical problems and theorems. This can potentially contribute to making mathematics more meaningful to a lot of students that are not destined to further pursue “pure” mathematics. The Fermat point of a scalene triangle, i.e., the point that minimizes the sum of the distances to the vertices can easily be modeled by weights hanging in balance as shown in Fig. 15. Soap bubble geometry can also be

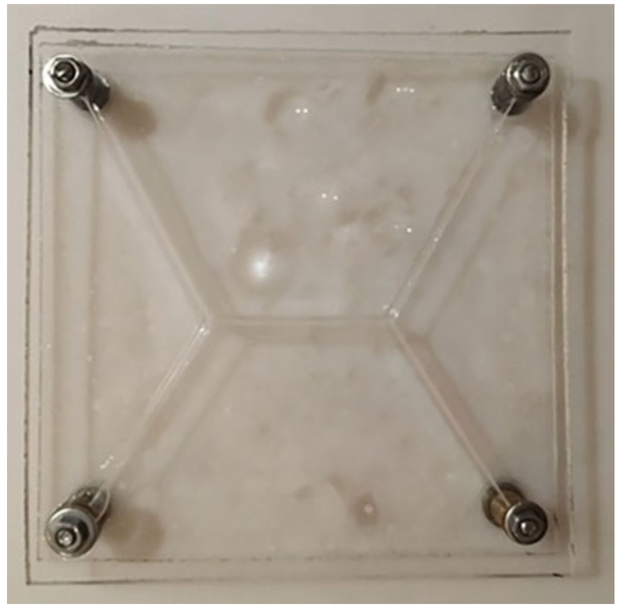


Fig. 16 Soap bubble demonstration of Steiner tree network for a square

used in class and workshops with teachers to quite dramatically illustrate the Fermat point of a scalene triangle, and as shown in Fig. 16, the similar Steiner shortest path network for a square.

7 Modelling the Real World

The inclusion of mathematical modelling as part of the mathematics curricula of many countries provides an excellent opportunity to include more experimentation into the classroom. As explained in Sect. 2, this is an important step beyond the Lakatosian point of view since, with modelling, experimenting with different hypotheses also comes into play.

There exist quite a lot of diagrammatic representations of the circular process of modelling (f. e. Blomhøj and Jensen 2003, p. 125; Kaiser and Stender 2013, p. 279). We use a very simplified diagram which consists of three important steps or stages as illustrated in Fig. 17, namely,

1) Construction of the mathematical model
2) Processing the model
3) Interpretation and evaluation of the results

Out of a practical problem, a mathematical model is constructed comprising a set of data and a set of hypotheses. The latter can be a geometric representation, a number of algebraic formulae or functions or differential equations, respectively, or a mixture of them. In the next step, the mathematical model is processed. This might mean different activities: to insert data into the equations, to manipulate formulae,