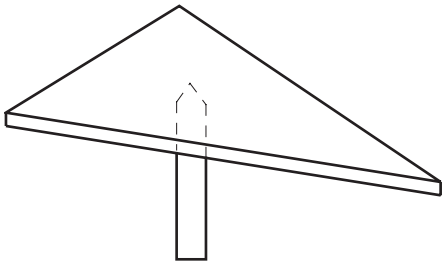
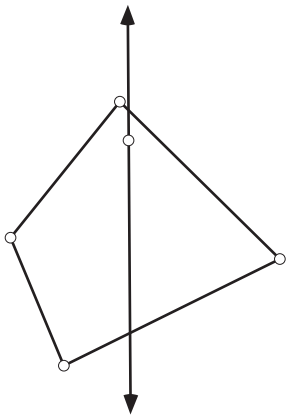
Name(s):

The Center of Gravity of a Triangle

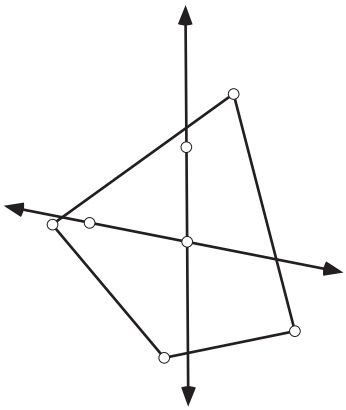


The balancing point of a two- or three-dimensional object is called its *center of gravity.* In architecture and engineering, accurately locating balancing points is extremely important for designing stable structures that do not collapse. You can locate the center of gravity of an object through experimentation. For example, when you balance a cardboard polygon on the tip of a pencil or an eraser, you have found the center of gravity of the polygon.

The pictures here show another way of locating the center of gravity of a cardboard polygon. The polygon is hanging on a string that is attached near its edge. The string is acting as a carpenter's plumb line, which provides the carpenter with a line perpendicular to the ground.



This quadrilateral is hanging by one point from a makeshift plumb line.

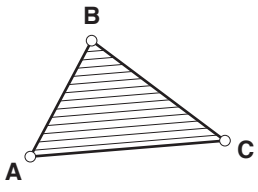


The same quadrilateral is hanging from a plumb line by a different point. The line of the previous plumb line is still showing.

The center of gravity is located where these two lines cross. Can you see why?

When you use experimentation to locate centers of gravity, the results are obviously subject to some experimental error.

In the following investigation, you will discover an accurate geometric way of locating the center of gravity of any triangle. The center of gravity of a triangle is also called a *centroid.* Imagine a triangle made up of thin horizontal beams, as shown here. Where is the center of gravity of each of these beams located? What can you conjecture about them?



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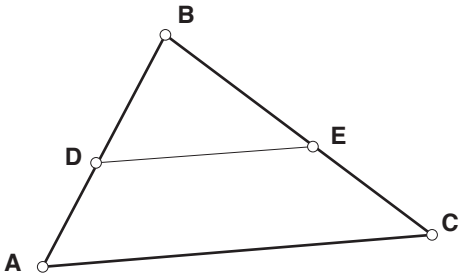
CONJECTURE: LOCATING THE CENTROID

- Open the sketch **Triangle Median.gsp.**

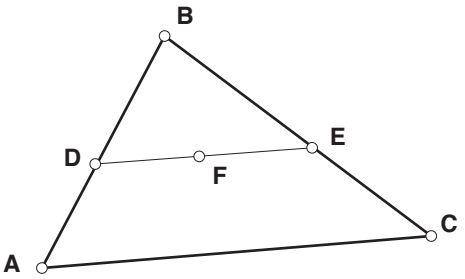


1. Drag point *D*. How is $\overline{DE}$ related to $\overline{AC}$?

2. Where is the center of gravity of $\overline{DE}$ located?

- Construct the center of gravity *F* of $\overline{DE}$.

3. Predict the path of *F* as *D* is dragged along $\overline{AB}$. Sketch your prediction in the triangle at right.



- Turn on tracing for point *F*. Now drag point *D* along $\overline{AB}$.

To trace point *F*, select *F* and choose **Trace Midpoint** from the Display menu. To turn off tracing, select *F* and again choose **Trace Midpoint** from the Display menu.

4. Describe the path of point *F*.

- Since the trace disappears when you click elsewhere, it is helpful to have a permanent *locus* of the path of *F*. Turn off tracing and construct this locus.

With point *F* still selected, also select point *D* and choose **Locus** from the Construct menu.

5. This path of point *F* is called a *median*. Briefly explain why the centroid of the whole triangle must lie somewhere on this median.

6. Write a definition (description) of a median.

7. Describe a shorter way to construct a median.

- Now use your method from Question 7 to construct the other two medians of the triangle.

8. Drag a vertex of your triangle. What can you say about the intersection of the three medians of a triangle?

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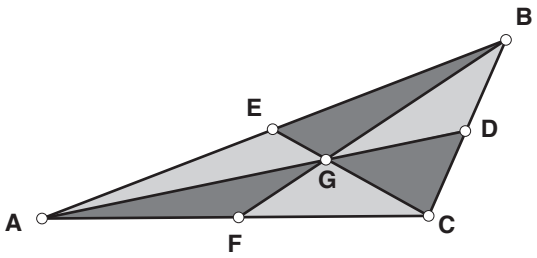
CONJECTURE: THE PROPERTIES OF THE CENTROID

In this investigation, you'll discover and prove some interesting properties of the centroid of a triangle.



- Open the sketch **Centroid.gsp** and drag a vertex to experiment with the sketch.

9. The medians of ΔABC are _____, _____, and _____. The centroid is _____.

- Show the distances along the medians. Also show the ratios of the distances.

10. Drag a vertex of your triangle. Write a conjecture about how the centroid G divides each median in a triangle.

- Hide the distances and their ratios and show the areas of the smaller triangles.

11. Drag a vertex and observe any relationships among the six areas. Write a conjecture about the areas of the smaller triangles formed by the three medians.

Although you are no doubt already convinced about your observations above, can you explain, in terms of other well-known geometric results, why your observations are true?

Mathematicians find the explanatory value of a logical argument useful. They also perceive the finding or development of a suitable explanation as an intellectual challenge. To them, it is as appealing as the solution of a complicated puzzle or a brain teaser, and as rewarding as the production of an original piece of music, art, or poetry. It could also be compared to the physical challenge of completing an arduous marathon or other physical task because it is a test of the intellectual ingenuity and stamina of the mathematician.

CHALLENGE Try to logically explain any of the conjectures you have made in this activity. After you have thought for a while and made some notes, use the hints that follow to develop an explanation of the three conjectures.

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In the preceding investigations you should have made three separate conjectures:

The three medians of a triangle always intersect in one point, and this point of concurrency is the centroid.

The centroid divides each median into the ratio 2:1.

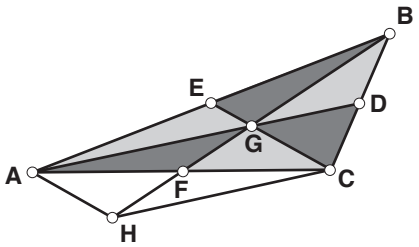
The six small triangles *AFG, CGF, GDC, BDG, GEB,* and *AGE* have equal areas.

EXPLAINING

You will begin by explaining the first two conjectures. Think through the sketch again. Segments $\overline{AD}$ and $\overline{CE}$ are medians intersecting at point *G*. Imagine joining *B* with *G* and extending this segment to *F* on $\overline{AC}$. We need to prove that *F* must always be the midpoint of $\overline{AC}$ (in other words, that $\overline{BF}$ is also a median and therefore all three meet in the same point *G*).

To extend $\overline{BF}$ this way, try using a circle and the **Ray** tool, or translate by a marked vector using commands from the Transform menu.

- Figure out a way to extend $\overline{BF}$ to *H* so that $\overline{GH} = \overline{GB}$. Drag a vertex of the original triangle to check your construction.
- Construct $\overline{HA}$ and $\overline{HC}$.



12. In ΔABH, what can you say about $\overline{EG}$ in relation to $\overline{AH}$? Why?

13. In ΔCBH, what can you say about $\overline{DG}$ in relation to $\overline{CH}$? Why?

14. Using Questions 12 and 13, what kind of quadrilateral is *AHCG*? Drag points to test your conclusion.

15. From Question 14, what can you now conclude about the diagonals $\overline{AC}$ and $\overline{GH}$ of quadrilateral *AHCG*?

16. What can you conclude about *F*?

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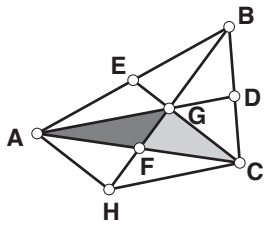
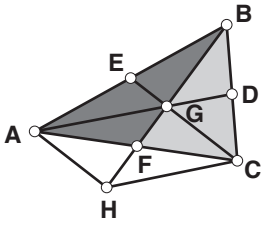
17. From Question 16, what can you now also say about $\overline{FG}$ in relation to $\overline{GB}$? Why?

Next, you'll explain why the six triangles have equal areas.

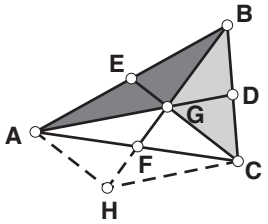


18. Consider the areas of the following pairs of adjacent triangles: *AFG* and *CFG* (shown in the picture), *CDG* and *BDG*, and *BEG* and *AEG*. Why are their areas equal?



19. What can you say about the areas of triangles *AFB* and *CFB* (shown in the picture)? Why?

20. From Questions 18 and 19, what can you conclude about the areas of triangles *ABG* and *CBG*? Why?



21. What can you now conclude about the areas of triangles *CDG, BDG, BEG,* and *AEG*? Why?

22. Now complete the logical explanation on your own.

Present Your Explanation

Write full explanations of your three original conjectures. You can use Questions 12–22 to help you. You may write your explanation as an argument in paragraph form or as a two-column proof. Use the back of this page, another sheet of paper, a Sketchpad sketch, or some other medium.

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Further Exploration

1. a. Can you find a way using geometric constructions in Sketchpad to locate the center of gravity of any "cardboard" quadrilateral?

b. Check your method in Question 1a with a concave quadrilateral. What do you notice about the center of gravity of a concave quadrilateral?

c. Try to generalize your method in Question 1a to any polygon.

2. Track and field star Dick Fosbury changed the high jump event forever by popularizing the backward-flip approach that came to be known as the "Fosbury flop." Fosbury was a student at Oregon State University (1965–69) when he used the "flop" to set an Olympic record with a jump of 7 ft $4\frac{1}{2}$ in. in Mexico City on October 21, 1968. Much ridiculed at the time, Fosbury's "flop" method eventually replaced the former standard "straddle" and "scissors" methods to become the standard method of world-class high jumpers. Why has this method proved to be more effective? (*Hint:* It has something to do with the center of gravity.)

3. You can locate the centroid of a triangle using coordinate geometry.

a. Measure the coordinates of the vertices of your triangle as well as its centroid. Calculate the average of the coordinates of the vertices. What do you notice? Check your observation by further dragging.

b. Assume that the vertices of a triangle have coordinates (x_1, y_1), (x_2, y_2), and (x_3, y_3). Show that the centroid is located at the "average" coordinates

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$

c. Use Sketchpad to find the "average" coordinates of the vertices of a quadrilateral and plot the point with those coordinates. Compare this point with the center of gravity of a "cardboard" quadrilateral as constructed in Question 1. What do you notice?

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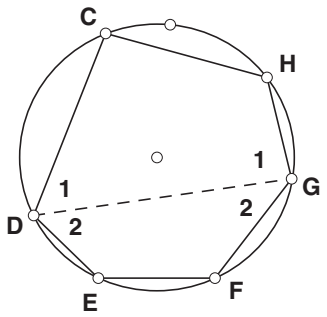

3. Consider the figure shown above. $m\angle C + m\angle G_1 = 180°$, and $m\angle E + m\angle E + m\angle G_2 = 180°$. Therefore, $m\angle C + m\angle E + m\angle G = 360°$. Similarly, it can be shown that the sum of the measures of the other set of alternate angles is also 360°.
4. Angles *A* and *C,* and angles *B* and *D,* are the two sets of alternate angles. The two sums of the measures of the sets of alternate angles of a cyclic quadrilateral are equal.
5. In general, for certain types of cyclic $2n$-gons where $n > 1$, the two sums of the measures of the sets of alternate angles are equal. (For convex ones, these sums are equal to $180°(n - 1)$.) The following theorem in this regard is proved in de Villiers (1996, 183–187):

   If $A_1A_2 \ldots A_{2n}$ $(n > 1)$ is any cyclic 2n-gon in which vertex $A_1 \rightarrow y\ A_{1+k}$ (vertex A_i is joined to A_{i+k}), the two sums of alternate interior angles are each equal to $m\pi$ (where $m = n - k$ and k is the total turning we would undergo by walking around the perimeter of the polygon).

THE CENTER OF GRAVITY OF A TRIANGLE
(PAGE 51)

This activity introduces students to the idea that even after a result has been found to be true experimentally, creating a logical explanation for the result can be an intellectual challenge similar to the solution of a crossword or other puzzle. Experience with students has shown that they find it far more reasonable to accept that some people (for example, mathematicians) could be motivated by such intellectual challenges and activities than to accept that they would indulge in such an activity simply for the sake of verification.

You could further point out that different people have different interests. For example, not everybody is excited by bungee jumping, mountaineering, marathon running, crossword puzzles, athletics, golf, bowling, cooking, tennis, or any given activity. This does not mean that someone who is not strongly motivated in any of these areas could not have some appreciation and respect for those who have mastered, and found meaning in, a particular discipline. The challenge is therefore to inculcate and encourage some appreciation and understanding of the discipline of mathematics (and in particular of deductive reasoning) in those who do not aspire to become mathematicians or who will not seriously apply mathematics in later life.

Prerequisites: Midpoint triangle theorem, properties of parallelograms, triangle area formula.

Sketches: Triangle Median.gsp and **Centroid.gsp.** Additional sketches are **Quad Centroid.gsp, Ceva.gsp, Ceva Concurrency.gsp,** and **Ceva Pentagon.gsp.**

CONJECTURE: LOCATING THE CENTROID

1. Segment *DE* is parallel to $\overline{AC}$.
2. The midpoint of $\overline{DE}$.
3. Responses will vary.
4. It is a straight line from the midpoint of $\overline{AC}$ to vertex *B.*
5. If the triangle is made up of many thin segments parallel to $\overline{DE}$, each of them will have its center of gravity along the path of point *F,* and therefore the

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center of gravity of the triangle as a whole should also lie somewhere along this path.

6. A median of a triangle is a line segment drawn from the midpoint of a side to the opposite vertex; or a median of a triangle is the path traced out by the center of gravity of a line segment moved parallel to one of its sides.
7. Connect the midpoint of a side to its opposite vertex.
8. The medians are always concurrent at the centroid.

CONJECTURE: THE PROPERTIES OF THE CENTROID

9. $\overline{AD}$; $\overline{BF}$; $\overline{CE}$; point G.
10. The centroid divides each median in the ratio 2:1.
11. The six small triangles all have the same area.

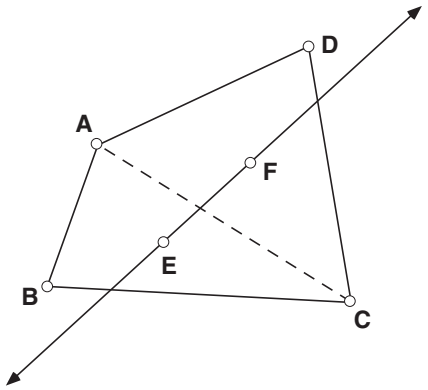
EXPLAINING

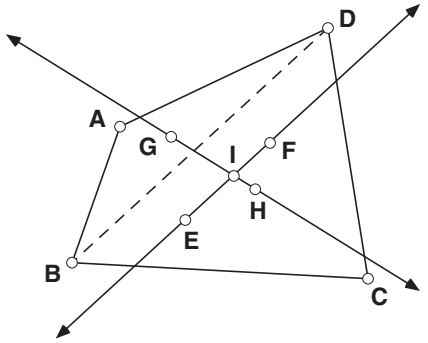
12. $\overline{EG} \parallel \overline{AH}$ (and $EG = \frac{1}{2}AH$), since E and G are the respective midpoints of sides AB and HB of ΔABH.
13. Similarly, $\overline{DG} \parallel \overline{CH}$ (and $DG = \frac{1}{2}CH$).
14. Therefore, $AHCG$ is a parallelogram (opposite sides parallel (and equal)).
15. Diagonals AC and GH bisect each other (property of parallelograms).
16. Therefore, F bisects $\overline{AC}$, and $\overline{BF}$ is also a median.
17. $FG = \frac{1}{2}HG$, but $HG = GB$ (construction), and therefore $FG = \frac{1}{2}GB$; that is, point G divides the median FB in the ratio 2:1. Similarly for the other medians.
18. Equal bases and heights.
19. Area AFB = area CFB (same bases and heights).
20. Area ABG = area CBG (area AFB − area AFG = area CFB − area CFG).
21. The areas of triangles CDG, BDG, BEG, and AEG are all equal.
22. Similarly, area ABG can be shown to be equal to area ACG, and therefore all six small triangles have the same area. (Students should be informed that although it may be good practice to write out the argument showing area ABG equal to area ACG in full, it is common mathematical practice, for the sake of brevity, not to do so.)

Present Your Explanation

This section provides students with the opportunity to organize and present their explanations clearly.

Further Exploration





1. a. The center of gravity of a cardboard quadrilateral $ABCD$ can be located by first constructing the centroids E and F of triangles ABC and ADC (see above). The center of gravity of the quadrilateral as a whole must now lie on the line connecting E and F. Next, construct the centroids G and H of triangles ABD and BDC. As before, the center of gravity of the quadrilateral as a whole must now lie on the line connecting G and H. Therefore, the center of gravity is located at I, namely, where the two lines EF and GH intersect.

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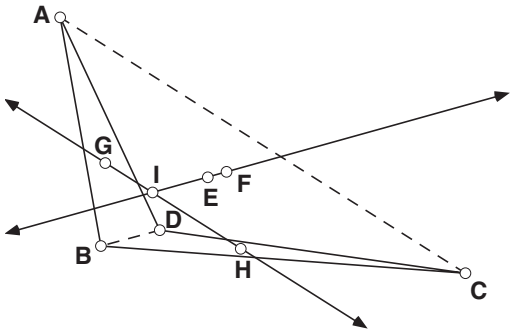


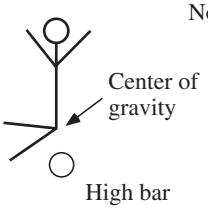
b. The center of gravity *I* of a concave cardboard quadrilateral can sometimes fall outside it, as shown in the figure. (In order to balance such a figure, one would have to attach its center of gravity to the figure with thin wires or rods—compare a wheel with spokes.)

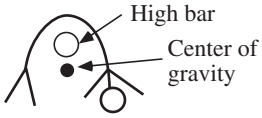
c. The center of gravity of a cardboard pentagon can be found by dividing it into a quadrilateral and a triangle and drawing a line through their respective centers of gravity. Next, divide it into any other quadrilateral and triangle, and again draw the line connecting their respective centers of gravity. The center of gravity of the pentagon as a whole is then located where these two lines meet.

This method can be generalized to any polygon by subdividing it into two paired sets of polygons of lower order. If two lines are then drawn connecting the respective centers of gravity of each pair of polygons, the center of gravity of the polygon as a whole is determined by the intersection of these two lines. (To locate the centers of gravity of higher-order polygons in Sketchpad, it is useful to build up a series of custom tools as you progress from quadrilaterals to pentagons, etc.)

2. The reason the "Fosbury flop" is most effective can be explained in terms of geometry. Suppose, for example, that a high jumper has enough energy to vertically displace his or her center of gravity by 7 ft. Then, using the traditional "scissors" and "straddle" techniques, this high jumper would have to move his or her center of gravity over the high bar, implying that the jumper would probably not be able to jump much higher than 6 ft 10 in. (see first figure). However, using the "Fosbury flop," the body assumes a shape (similar to a concave figure) whose center of gravity actually moves outside the body. In other words, if the same high jumper used the "Fosbury flop," the jumper's center of gravity could pass under the bar while his or her body went over it (see second figure). Using this technique, the same high jumper could therefore possibly jump 7 ft 2 in.—a gain of 4 inches: enough for a gold medal and a new Olympic record!

Not drawn to scale





3. a. The average value of abscissa and ordinates of the vertices are equal to the value of the abscissa and ordinate of the centroid.

b. Note that generally speaking, analytic (coordinate) proofs tend to be less *explanatory* than synthetic geometry proofs, and therefore tend more to serve a *verification* function.

Consider the following figure. The midpoint of $\overline{AB}$ has coordinates $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$. We shall now use the useful result from coordinate geometry that a point that divides a segment AB with coordinates $A(x_1, y_1)$ and $B(x_2, y_2)$ into the ratio $m:n$ has coordinates

$$\left(x_1\left(\frac{n}{m+n}\right) + x_2\left(\frac{m}{m+n}\right), y_1\left(\frac{n}{m+n}\right) + y_2\left(\frac{m}{m+n}\right)\right)$$

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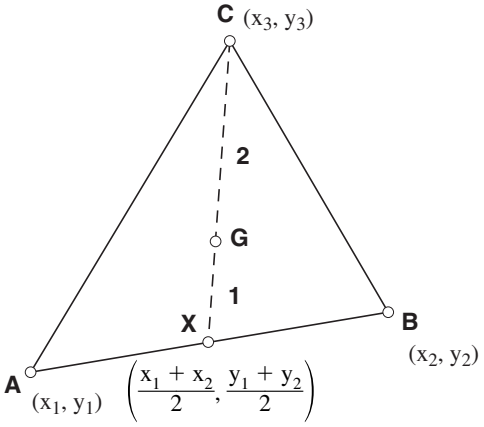
Since the centroid divides segment *CX* into the ratio 2:1, the coordinates of the centroid can be determined as follows:

$$\left(x_3\left(\frac{1}{3}\right) + \left(\frac{x_1 + x_2}{2}\right)\left(\frac{2}{3}\right), y_3\left(\frac{1}{3}\right) + \left(\frac{y_1 + y_2}{2}\right)\left(\frac{2}{3}\right)\right)$$

$$= \left(\frac{2x_3 + 2x_1 + 2x_2}{6}, \frac{2y_3 + 2y_1 + 2y_2}{6}\right)$$

$$= \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$$



Another approach is to assume that (x_1, y_1) is placed at $(0, 0)$ and (x_2, y_2) at $(x_2, 0)$. Then, by determining the equations of two of the medians, we can locate the centroid by solving this pair of simultaneous equations. This approach, however, involves quite tedious algebra.

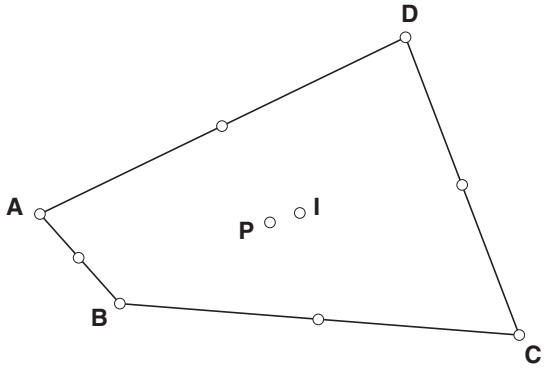
c. Note that averaging the coordinates is equivalent to finding the center of gravity of three point masses with equal weight, situated at the vertices. Although this averaging method is easily generalized, it does *not* provide the center of gravity of a cardboard polygon (except for a triangle), but the center of gravity of equal point masses placed at the vertices. For example, the following diagram shows the center of gravity *I* of a "cardboard" quadrilateral. If, however, only point masses of equal weight are placed at the same vertices, their center of gravity is located at *P,* which clearly does not coincide with *I.*

A: (–8.184, –0.635)
D: (–0.212, 3.210)
C: (2.222, –3.281)
B: (–6.421, –2.611)

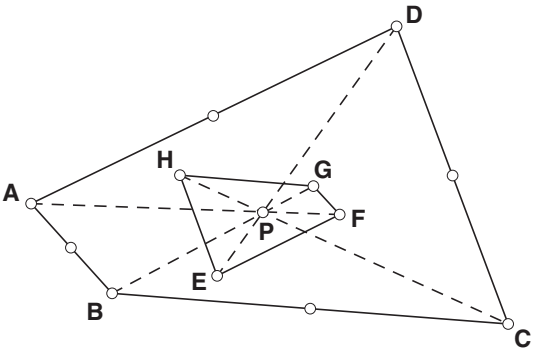
$\frac{x_B + x_C + x_D + x_A}{4} = -3.149$

$\frac{y_B + y_C + y_D + y_A}{4} = -0.829$

P: (–3.149, –0.829)
I: (–2.485, –0.636)



A purely geometric method for locating the point *P* is shown in the next diagram (from **Quad Centroid.gsp**). Determine the respective centroids *E, F, G,* and *H* of triangles *ABC, BCD, CDA,* and *DAB.* Then quadrilateral *EFGH* is similar to quadrilateral *DABC,* with a reduction factor of 3, and the lines connecting corresponding vertices are concurrent at the center of the similarity, which is the center of gravity of equal point masses placed at the vertices. A proof of the similarity of these two quadrilaterals and the concurrency of the lines can be found in Yaglom (1962, 21) or de Villiers (1996, 189).



Alternative Explanation for Median Concurrency

An alternative explanation will now be given; you could present it to students to illustrate the discovery function of proof, since it leads to an immediate generalization

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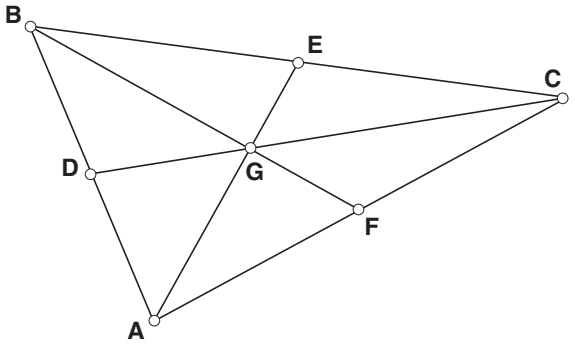
of the result, namely Ceva's theorem. (See **Ceva.gsp** and **Ceva Concurrency.gsp.**) Some students may have difficulty in step 2 understanding the reasoning involving the application of the *dividendo* rule. It may help to give students some practical examples, such as $\frac{9}{12} = \frac{3}{4} = \frac{9-3}{12-4} = \frac{6}{8}$, and so on.

Consider again this conjecture: *The three medians of any triangle always meet in one point (are concurrent).*



1. Let $\overline{AE}$ and $\overline{CD}$ be medians intersecting at point *G*. Join *B* with *G* and extend to *F* on $\overline{AC}$. We must now show that *F* is the midpoint of $\overline{AC}$ (in other words, that $\overline{BF}$ is also a median and therefore that all three meet in the same point *G*).

2. If we denote the area of triangle *ABC* by the notation (*ABC*), we have

$$\frac{(BAF)}{(BFC)} = \frac{\frac{1}{2}h_1 AF}{\frac{1}{2}h_1 FC} = \frac{AF}{FC} \text{ and } \frac{(GAF)}{(GFC)} = \frac{\frac{1}{2}h_2 AF}{\frac{1}{2}h_2 FC} = \frac{AF}{FC}$$

Therefore, $\frac{AF}{FC} = \frac{(BAF)}{(BFC)} = \frac{(GAF)}{(GFC)}$

$$= \frac{(BAF) - (GAF)}{(BFC) - (GFC)} = \frac{(BAG)}{(BCG)} \cdots$$

dividendo.

Similarly, we find $\frac{CE}{BE} = \frac{(ACG)}{(BAG)}$ and $\frac{BD}{AD} = \frac{(BCG)}{(ACG)}$.

3. But it is given that $BE = EC$ and $AD = DB$. Therefore, $(BCG) = (ACG)$ and $(ACG) = (BAG)$, which implies $(BAG) = (BCG)$. But the areas of these two triangles are proportional to *AF* and *FC,* as shown by the second equation. Thus, $\frac{AF}{FC} = 1$ implies $AF = FC$ and completes the explanation.

Looking Back

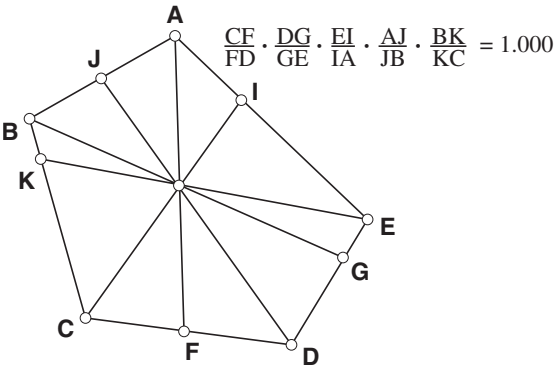
Now encourage your students to look back carefully at the proof. Let them consider the product of the three ratios $\frac{AF}{FC}$, $\frac{AD}{DB}$, and $\frac{BE}{CE}$ expressed in terms of areas in step 2. Ask them what they notice about this product, and whether in deriving these three ratios the properties that *E* and *D* are midpoints were used at all. Further, ask them what they can therefore conclude from this.

You can then point out that

$$\frac{AF}{FC} \times \frac{AD}{DB} \times \frac{BE}{CE} = \frac{(BAG)}{(BCG)} \cdot \frac{(ACG)}{(BAG)} \cdot \frac{(BCG)}{(ACG)} = 1$$

and that the properties that *E* and *D* are midpoints were not used at all in this derivation. Therefore, we can immediately generalize that if, in any triangle, $\times$ line segments *AD, BF,* and *CE* are concurrent (with *D, F,* and *E* respectively on sides *BC, AC,* and *AB*), then $\frac{AF}{FC} \times \frac{AD}{DB} \times \frac{BE}{CE} = 1$. The converse of this result is also true and can be proved by using proof by contradiction.

This interesting result is called Ceva's theorem, after an Italian mathematician named Giovanni Ceva, who published it in 1678. In his honor, the line segments *AE, BF,* and *CD* joining the vertices of a triangle to any given points on the opposite sides are called *cevians.* (Note that apart from the medians, the altitudes and angle bisectors of a triangle can be considered cevians if they are extended to meet the opposite sides.) Although it is not known exactly how Ceva discovered this result, it is likely that he discovered it logically in a fashion similar to that outlined above, and not by using construction and measurement.



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Interestingly, Ceva's theorem can also be generalized to polygons (see Grünbaum and Shepherd 1995), and you may wish to demonstrate it to your students. (See **Ceva Pentagon.gsp.**) However, for polygons with an even number of sides, this generalization does not deal with the ratios into which the cevians divide the opposite sides, but rather with the ratios into which the cevians divide the diagonals.

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