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Centroid of a Polygon—Three Views

When they investigate medians of a triangle, geometry students are often challenged to balance a cardboard model on the tip of a pencil. Of course, they find that the eraser end works much better than the tip. The point of balance for the cardboard model is referred to in physics as the *center of mass*, the point at which the entire mass of the model appears to be concentrated. Having experienced the center-of-mass concept physically in this type of exercise, students may be ready for a mathematical look. We present a possible excursion here.

Before we get into the details, let us distinguish among some terms. First, what is a *polygon?* We like the following definitions:

Definition: "A *polygon* is the union of three or more segments in the same plane such that each segment intersects exactly two others, one at each of its endpoints" (Coxford 1993, p. 92). The segments are the *sides* and the endpoints of the segments are the *vertices*.

Definition: "The union of a polygon and its interior is a *polygonal region*" (Coxford 1993, p. 94).

Geometry students are often challenged to balance a cardboard triangular region on a pencil tip

Figure 1 shows some polygon-related sets, with the vertices subscripted consecutively for later convenience. **Figure 1a** shows a convex quadrilateral. **Figure 1b** shows a nonconvex pentagon. **Figure 1c** shows the polygonal region corresponding to the polygon in **figure 1a. Figure 1d** shows only the *vertex set* for the polygon in **figure 1b.**

When we investigate the idea of the *center of mass* of a polygon, we must distinguish among a *polygon,* which is modeled by a wire frame, as in **figures 1a** and **1b;** a polygon's *vertex set,* modeled by a set of dots, as in **figure 1d;** and a *polygonal region,* modeled by a piece of sheet metal or cardboard, as in **figure 1c.** We look first at polygonal regions, since one thinks of these regions when the phrase "centroid of a polygon" arises.

Proving that the medians of a triangle meet in a point, commonly referred to as the *centroid of the triangle,* is standard fare in geometry classes. That this point is also the center of mass of the corresponding triangular region of uniform mass density is usually left to the intuitive balancing act mentioned previously. The purpose of this article, in the spirit of exploration and generalization, is to move to the question, What would we mean by the center of mass, to be referred to as a *centroid* here, of a polygonal region with more than three sides? And if we can accept its existence, what can we say about its location? Thinking of the cardboard-triangle demonstration, students might naturally want to try another balancing act. This approach is valid in theory but difficult in practice.

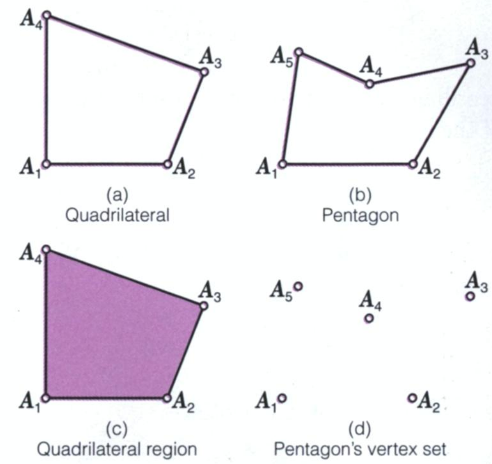

Fig. 1
Polygon-related sets

If no student mentions the plumb-line technique for finding the centroid of a rigid polygonal sheet, the teacher can share with the class a result that carpenters and others have long known—suspending the sheet from a vertex and attaching a weighted string from the same vertex causes the string to pass through the centroid. Two such suspensions will determine the centroid **(fig. 2).** The suspensions

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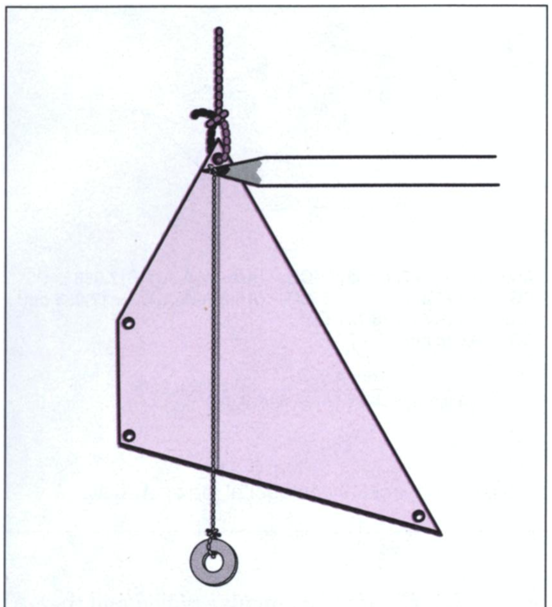

Fig. 2
Using the plumb-line method to find the physical centroid of a quadrilateral region

need not be from vertices, but suspension points distant from the centroid give the best accuracy.

Having physically established the existence of the centroid of a rigid uniform polygonal sheet, the teacher can launch into a rather nice example of recursion and give some mathematical substance to the physical concept. In this venture, the use of interactive software is very helpful. The figures here were made using The Geometer's Sketchpad (Jackiw 1995).

Let us first look at a triangular region. In **figure 3,** the median A_3M_3 divides the triangular region $A_1A_2A_3$ into two parts with equal areas because triangles $A_1A_3M_3$ and $A_2A_3M_3$ have equal bases, A_1M_3 and A_2M_3, and a common altitude from A_3, hence the same area. We can easily show that the centroid G of triangle $A_1A_2A_3$ is midway between centroids Q and R of triangles $A_1A_3M_3$ and $A_2A_3M_3$. Thus, the products (GQ)(area $A_1A_3M_3$) and (GR)(area $A_2A_3M_3$) are equal. Since the mass of region $A_1A_3M_3$ is concentrated at Q, whereas the mass of region $A_2A_3M_3$ is concentrated at R, the equality of the products illustrates the familiar teeter-totter principle. The notion of balancing subregions by suitably placing the balance point will be exploited as we generalize.

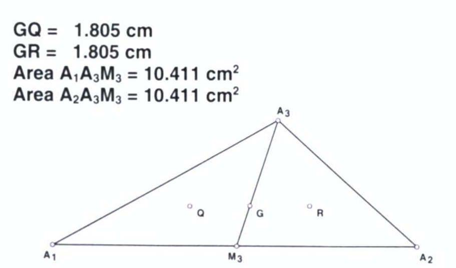

Fig. 3
G is the centroid of triangular region $A_1A_2A_3$.

We next turn to a quadrilateral region and use what we know about triangular regions to construct the centroid of the four-sided region. In this endeavor, it is useful to employ a script in The Geometer's Sketchpad to construct the centroids of the four triangular regions determined by the diagonals of the quadrilateral. After completing a construction, exercising the Make Script option under Work in the toolbar saves the selected steps for later use. In **figure 4,** these points are G_1, G_2, G_3, and G_4, the centroids of triangular regions $A_2A_3A_4$, $A_3A_4A_1$, $A_4A_1A_2$, and $A_1A_2A_3$, respectively. Following Peterson (1997), we construct C as the common point of G_1G_3 and G_2G_4. Assuming uniform density of the material in a model, Sketchpad's calculations shown in the figure indicate that C is the balance point of the opposing pairs of triangular regions. Dragging a vertex A_i gives strong evidence that the method for determining C is valid, since the products remain equal.

Suspension points distant from the centroid give the best accuracy

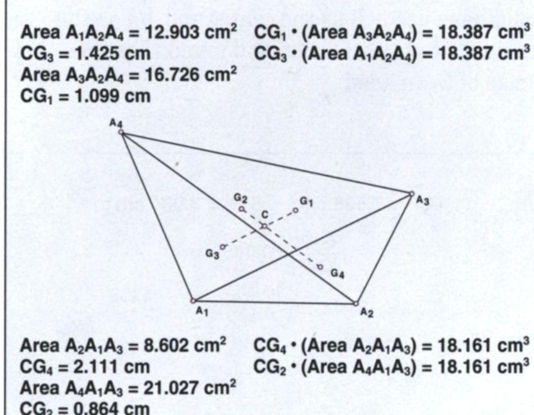

Fig. 4
C is the centroid of quadrilateral region $A_1A_2A_3A_4$.

Although this approach of decomposing a quadrilateral region into a disjoint union of two triangular regions was successful in finding the centroid of the quadrilateral region, it does not seem to extend to the next level, since a pentagonal region has five triangles that can be formed from three consecutive vertices. Which ones should be chosen in pairs to form segments as we did with the quadrilateral's four centroids? Because of the limitations of this method, we look for an alternative.

The equality of the products reinforces the correctness of the dilation method

Examining the numbers above the drawing in **figure 4,** in particular, the equations $(CG_1) \cdot (\text{area } A_3A_2A_4) = 18.387 = (CG_3)(\text{area } A_1A_2A_4)$, we see that

$$\frac{CG_1}{CG_3} = \frac{\text{area } A_1A_2A_4}{\text{area } A_3A_2A_4}.$$

We add 1 to both sides of this equation to get

$$\frac{CG_1 + CG_3}{CG_3} = \frac{\text{area } A_1A_2A_4 + \text{area } A_3A_2A_4}{\text{area } A_3A_2A_4},$$

or

$$\frac{G_1G_3}{CG_3} = \frac{\text{area } A_1A_2A_3A_4}{\text{area } A_3A_2A_4}.$$

We write the reciprocals as

$$\frac{CG_3}{G_1G_3} = \frac{\text{area } A_3A_2A_4}{\text{area } A_1A_2A_3A_4}.$$

Calling the right side of the last equation *k,* which here is 0.565, we use a dilation with center G_3 and scale factor k and apply it to G_1. The image of G_1 is then the desired centroid *C.* Recall, for example, that the dilation with center G_3 and scale factor 2/5 pulls every point in the plane back toward G_3 so that the distance from G_3 to the image is two-fifths of the distance from G_3 to the original point. **Figure 5** illustrates this effect, where P' is the image of P under the dilation $D(G_3, 2/5)$. Note that we could have used G_1 as the center and $1/k$ as the scale factor of the dilation and found C as the image of G_3 instead.

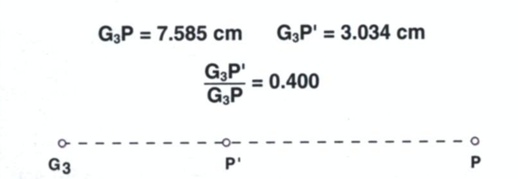

Fig. 5
The dilation with center G_3 and scale factor 2/5 maps P to P'.

Readers can verify that the same value of k is obtained if the numbers below the drawing in **figure 4** are used instead. The upshot is that an appropriate dilation locates the centroid *C,* using just two centroids of subregions. We save the construction of C as a script.

Let us now move on to find the centroid of a pentagonal region $A_1A_2A_3A_4A_5$, making use of our techniques for simpler regions. In **figure 6,** we decompose this region into the triangular region $A_1A_2A_3$ and the quadrilateral region $A_3A_4A_5A_1$. We employ two Sketchpad scripts, one to find the centroid, call it G_1, of the triangular region and the other to find the centroid, call it G_5, of the quadrilateral region. The dilation with center G_5 and scale factor 0.229, the ratio of

$$\frac{\text{area } A_1A_2A_3}{\text{area } A_1A_2A_3 + \text{area } A_3A_4A_5A_1},$$

sends G_1 to the centroid C of the pentagonal region. As a check, we copy the same region, minus some of the clutter, as **figure 7,** where we use point C from **figure 6** with triangular region $A_1A_4A_5$, whose centroid is G_4, and quadrilateral region $A_1A_2A_3A_4$, whose centroid is G_3. The equality of the products shown in **figure 7** reinforces the correctness of the dilation method, based on the teeter-totter principle, employed to find C.

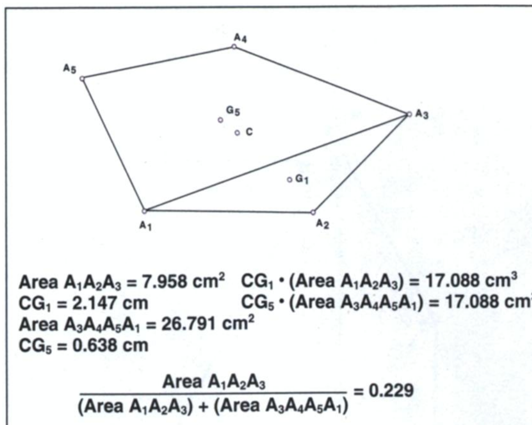

Fig. 6
C is the centroid of pentagonal region $A_1A_2A_3A_4A_5$.

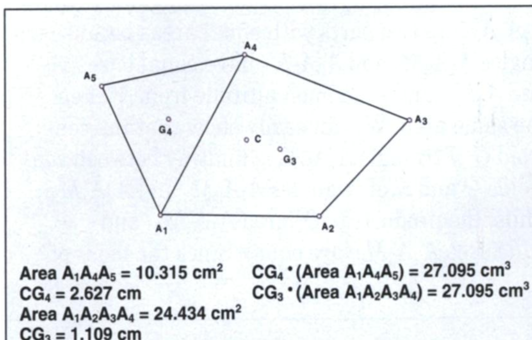

Fig. 7
Finding the centroid, C, of the pentagonal region from **figure 6** by a different decomposition

Saving the construction of the pentagonal region's centroid as a Sketchpad script permits the construction of the centroid of a hexagonal region, which is not shown, and so on. Interestingly, but not surprisingly, the decomposition of a hexagonal region need not be into a triangular and a pentagonal region, as the recursive method used here dictates, but can instead employ two nonoverlapping quadrilateral regions with the same result.

VERTEX SETS

We next look at the triangle from a different viewpoint, focusing on the three vertices only and not on the segments or the interior. This view is sometimes called a *mass points approach.* (See Skinner [1995] for an example.) This approach uses the idea that three masses m_1, m_2, and m_3 located at noncollinear points P_1, P_2, and P_3 are equivalent to, and can be replaced by, a single mass $m_1 + m_2 + m_3$ located at the "center of mass" of triangle $P_1P_2P_3$. The center of mass will be the geometric centroid, or common point of the medians, if and only if $m_1 = m_2 = m_3$.

Applying this idea to a triangle $A_1A_2A_3$ with unit masses at the vertices, we replace the two masses located at A_1 and A_2 with a single mass of two units located at midpoint M_3 **(fig. 8).** The balance point G of segment M_3A_3 is two-thirds of the way from A_3 to M_3, and again, we see the teeter-totter principle at work. The entire mass, or 3 units, of the original configuration can then be replaced by a three-unit mass located at G. Symmetry and the concurrence property of a triangle's medians imply that we can choose any two vertices to consider when we do the mass replacement. We note that the dilation $D(G, -2)$ sends M_i to A_i, for $i = 1, 2, 3$, where a negative scale factor means that the center G of the dilation is between the original point M_i and its image A_i and where $GA_i = 2(GM_i)$.

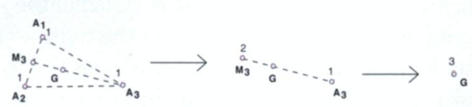

Fig. 8
G is the centroid of the vertex set $\{A_1, A_2, A_3\}$.

Let us now look at a quadrilateral's vertex set. Each vertex A_i of a quadrilateral $A_1A_2A_3A_4$ corresponds to a triangle formed by the remaining three vertices, and each of these triangles has a centroid G_i. **Figure 9** focuses on centroid G_4. Here the three unit masses at A_1, A_2, and A_3 are replaced by one three-unit mass at G_4, and finally the three units at G_4 and the one unit at A_4 are replaced by four units at Q, where Q is on A_4G_4 and $QA_4 = 3(QG_4)$.

In **figure 10,** we show the four centroids G_i and observe that the point Q inside quadrilateral $A_1A_2A_3A_4$ lies on the four segments joining the vertices to the corresponding centroids G_1, G_2, G_3, and G_4 of its four triangles. The dilation $D(Q, -3)$ sends

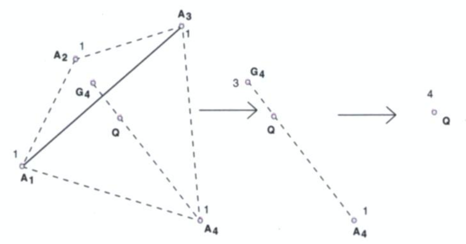

Fig. 9
Q is the centroid of the vertex set $\{A_1, A_2, A_3, A_4\}$.

the centroids of the four triangles to the corresponding vertices of the quadrilateral. That is, $D(Q, -3) \cdot (G_i) = A_i$, $i = 1, 2, 3, 4$. In other words, $A_iQ = (3/4)(A_iG_i)$. We argue as justification that unit masses at the three points A_2, A_3, and A_4 can be replaced by one three-unit mass at G_1 and that Q is the balance point of segment A_1G_1, where A_1 has a one-unit mass, by the teeter-totter principle. The symmetry of the argument and the uniqueness of the centroid from physical considerations imply that the same relation holds for the other A_i, G_i pairs. A somewhat tedious vector proof that for each i, $\overrightarrow{A_iQ} = (3/4)\,\overrightarrow{A_iG_i}$ is also possible and is available from the authors. **Figure 11** shows the analogous result for the vertex set of a pentagon; P is the centroid of the five-vertex set. **Figure 11** was constructed from a script of the construction of Q in **figure 10.** The general result for vertex sets can be stated as follows:

> Let $\{A_1, A_2, \ldots, A_n\}$ be the vertex set of a polygon of n sides. Let G_k be the centroid of the vertex set of the polygon formed by omitting vertex A_k. The segments A_iG_i are concurrent at a point C, the centroid of the vertex set $\{A_1, A_2, \ldots, A_n\}$, and for each $i = 1, 2, \ldots, n$, we have $A_iC = ([n - 1]/n)A_iG_i$.

Again, we see the teeter-totter principle at work

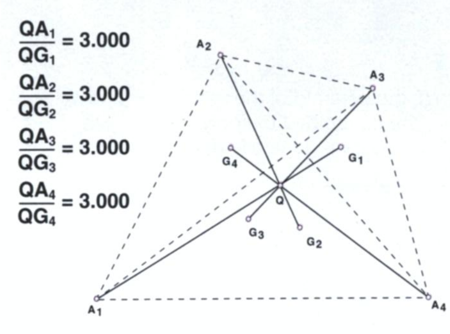

Fig. 10
Q, the centroid of the vertex set $\{A_1, A_2, A_3, A_4\}$, lies on the four segments A_iG_i, $i = 1, 2, 3, 4$.

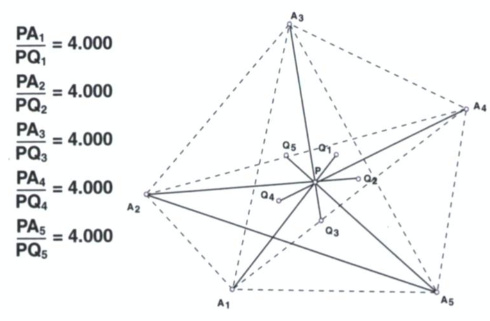

Fig. 11
P, the centroid of the vertex set $\{A_1, A_2, A_3, A_4, A_5\}$, lies on the five segments A_iG_i, $i = 1, 2, 3, 4, 5$.

An alternative formulation makes the centroid of the vertex set computationally much easier by using vectors and coordinates. With n vertices, C as the centroid, and O an arbitrarily chosen origin from which to draw vectors to the vertices, we have

$$\overrightarrow{OC} = \frac{1}{n}(\overrightarrow{OA_1} + \overrightarrow{OA_2} + \cdots + \overrightarrow{OA_n}).$$

To justify this result with $n = 4$, we use the notation of **figure 12** and take A_3 as O:

$$\overrightarrow{A_3G_3} = \overrightarrow{A_3A_1} + \overrightarrow{A_1A_4} + \overrightarrow{A_4G_3}$$

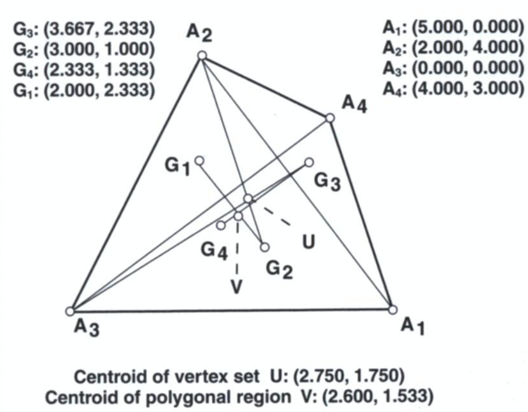

Fig. 12
U, the centroid of the vertex set, and V, the centroid of the polygonal region, are generally different.

Using the median property for triangles, we have

$$\overrightarrow{A_4G_3} = \frac{2}{3}\left\{\frac{1}{2}\overrightarrow{A_4A_2} + \frac{1}{2}\overrightarrow{A_4A_1}\right\}$$
$$= \frac{1}{3}\overrightarrow{A_4A_2} + \frac{1}{3}\overrightarrow{A_4A_1}.$$

Thus,

$$\overrightarrow{A_3G_3} = \overrightarrow{A_3A_1} + \overrightarrow{A_1A_4} + \frac{1}{3}\overrightarrow{A_4A_2} + \frac{1}{3}\overrightarrow{A_4A_1}$$
$$= \overrightarrow{A_3A_1} + \frac{2}{3}\overrightarrow{A_1A_4} + \frac{1}{3}\overrightarrow{A_4A_2}$$
$$= \overrightarrow{A_3A_1} + \frac{2}{3}(\overrightarrow{A_1A_3} + \overrightarrow{A_3A_4}) + \frac{1}{3}(\overrightarrow{A_4A_3} + \overrightarrow{A_3A_2})$$
$$= \overrightarrow{A_3A_1} + \frac{2}{3}\overrightarrow{A_1A_3} + \frac{2}{3}\overrightarrow{A_3A_4} + \frac{1}{3}\overrightarrow{A_4A_3} + \frac{1}{3}\overrightarrow{A_3A_2}.$$

Therefore,

$$\overrightarrow{A_3G_3} = \frac{1}{3}\overrightarrow{A_3A_1} + \frac{1}{3}\overrightarrow{A_3A_4} + \frac{1}{3}\overrightarrow{A_3A_2}.$$

Substituting for $\overrightarrow{A_3G_3}$, we get

$$\overrightarrow{A_3U} = \frac{3}{4}\overrightarrow{A_3G_3}$$
$$= \frac{1}{4}(\overrightarrow{A_3A_1} + \overrightarrow{A_3A_4} + \overrightarrow{A_3A_2} + \overrightarrow{A_3A_3}),$$

as asserted.

The reader can verify these computations by using the coordinates in **figure 12.** The primary reason for including **figure 12** is to show that for a quadrilateral, the centroid U of the vertex set is different from the centroid V of the polygonal region. These points coincide for triangles but not for higher polygons, except in special cases. Thus, the centroid concept has at least two interpretations—one for polygonal regions and one for their vertex sets—as well as a third, which follows.

POLYGONS

Let us now consider centroids of *polygons*, those figures modeled by wire frames. Again assuming uniform density of the sides, we replace each side by a mass proportional to its length and located at the midpoint of the side. Finding the centroid of the resulting figure is similar to finding the centroid of the set of midpoints of the sides, which results in a vertex set for a different polygon; but we now have to account generally for unequal masses at the midpoints. Using the A_i's from **figure 12** for illustration in **figure 13,** we have $A_3A_1 = 5$, so we put a mass of 5 at (2.5, 0). Similarly, we put a mass of 10, the length of A_1A_4, at the midpoint (4.5, 1.5) of A_1A_4; put a mass of 5 at the midpoint (3, 3.5) of A_4A_2; and put a mass of 20 at the midpoint (1, 2) of A_2A_3. With A_3 as the origin O and K as the centroid of the wire-frame quadrilateral, we use the alternative formulation above, modified to account for the different masses at the midpoints. Letting t represent the total mass,

$$t = 5 + \sqrt{10} + \sqrt{5} + \sqrt{20},$$

we get

$$\overrightarrow{OK} = \frac{5}{t}\langle 2.5, 0\rangle + \frac{\sqrt{10}}{t}\langle 4.5, 1.5\rangle + \frac{\sqrt{5}}{t}\langle 3, 3.5\rangle + \frac{\sqrt{20}}{t}\langle 1, 2\rangle,$$

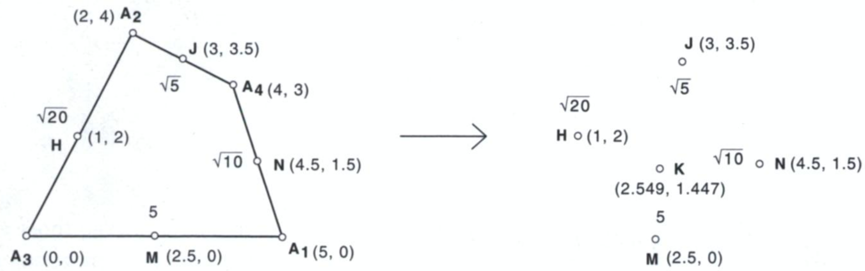

Fig. 13
K, the centroid of the wire-frame quadrilateral from **figure 12**, is different from centroids *U* and *V*.

which reduces to approximately <2.549, 1.447>. Thus, the centroid *K* of the polygon is (2.549, 1.447), which is different from both *U* and *V* in **figure 12.**

SUMMARY

We have seen that the phrase "centroid of a polygon" needs clarification, since for a given polygon, generally three different points are (1) the "centroids" of the corresponding polygonal region, (2) the vertex set of the polygon, and (3) the polygon itself. The authors are indebted to reviewers Eileen Schoaff, Carl Backman, Ronald Scoins, and Richard Muller for prompting a more careful look at these distinct ideas than we had originally presented. We have then seen that a blend of physics (the teeter-totter principle), mathematics (vectors, coordinate geometry, and recursion), and technology (for exploration and clarification) produces results that may be surprising, and these results serve to generalize the notion of "centroid" to polygons other than triangles.

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