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◈ THE ART OF TEACHING ◈

Centroids

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In geometry classes we occasionally meet with the question: "What is geometry good for?" Usually the description of practical applications will provide the necessary answer. But in some cases the same question if it had been more accurately formulated, would read: "Can you convey to me experiences showing that geometry has a meaning and reality beyond man-made definitions and theorems?" Then the answer could be given somewhat in the line of the following example. One would say to the student: "Let me show you something. Here is a piece of cardboard in shape of a triangle. Try to balance it on the eraser-end of your pencil." The student will make a few attempts and finally succeed in finding the point where the cardboard has to be supported in order to uphold itself in equilibrium. Then one will show that the same point can be obtained without any experimentation through geometric construction. It is the point of intersection of the medians of the triangle (Figure 1).

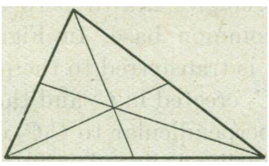

Fig. 1. Centroid of a triangle.

A study of centroids is especially fit to build the bridge between geometric constructions and facts related to natural sciences. The position of a centroid is independent of the circumstance whether one measured in inches or centimeters and of whatever terms or definitions had been used in support of the constructions.

From the centroid of a triangle one can proceed to the centroid of a quadrilateral. A diagonal divides a quadrilateral into two triangles. For each of the triangles the centroid can be obtained as the point of intersection of two medians. An edge supporting both centroids balances the quadrilateral; the straight line joining the centroids is a locus for the centroid of the quadrilateral (see Figure 2, left diagram). The second diagonal also divides a quadrilateral into two triangles which are different from the first ones. The straight line joining their centroids is again a locus for the centroid of the quadrilateral (see Figure 2, right diagram). Therefore the centroid of the quadrilateral is found as the point of intersection of the two loci (see Figure 2, third diagram). By means of cutting the given quadrilateral out of a piece of cardboard the position of the centroid can be checked experimentally.

The 8 medians through which the centroid of the quadrilateral has been obtained form a stellar octagon which is inscribed in the quadrilateral. Figure 3 shows a stellar octagon inscribed in a circle. The circle is divided into 8 equal parts and every point of division is joined with the third one following it in either way around the circle. Figure 4 shows a stellar octagon inscribed in a square. The 8 points which are used to construct the stellar octagon are the four vertices and the four middle points of the sides of the square. Each point is joined with the third one following it in either way around the square. In Figure 5 the construction of

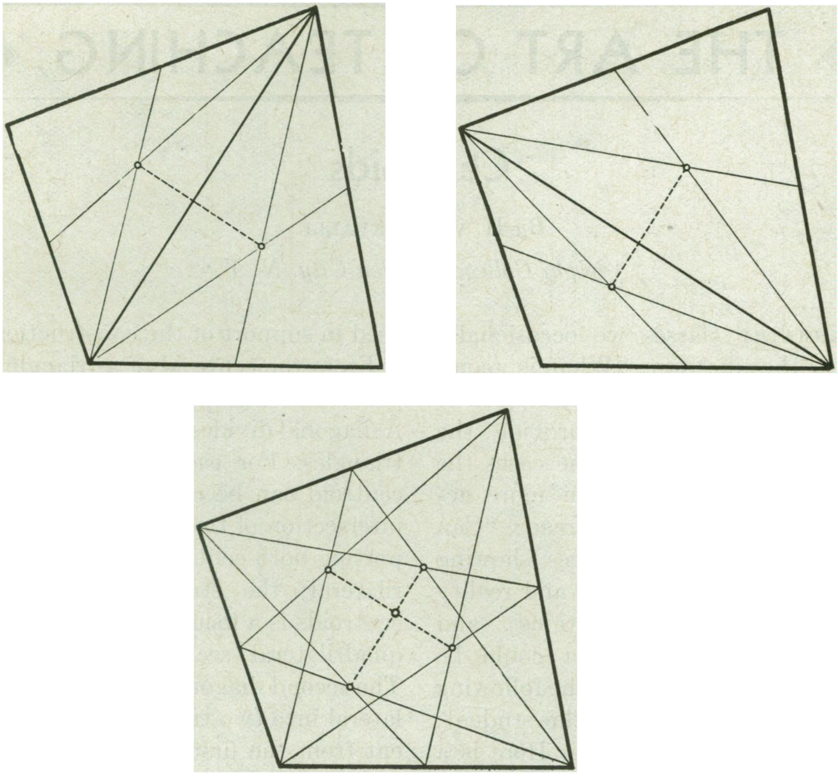

FIG. 2. Centroid of a quadrilateral.

an inscribed stellar octagon is repeated once more for an irregular quadrilateral, using again its vertices and the middle points of its sides. The dotted lines connecting the centroids of the partial triangles intersect in the centroid of the quadrilateral.

A short cut in the construction of the centroid of a quadrilateral can be achieved through the method shown in Figure 6. The given quadrilateral ABDE is divided through the diagonal BE into two triangles (ΔABE and ΔBDE). In each triangle two medians are drawn and their points of intersection C_1 and C_2 are the centroids. The straight line $C_1 C_2$ connecting the centroids contains the centroid of the quadrilateral. Would the two triangles have been of the same area their common centroid C would be half way between C_1 and C_2. But as soon as one triangle is larger than the other C lies closer to the centroid of the larger triangle. Following the law of moments the ratio of the distance of C from C_1 and C_2 is the reciprocal values of the ratio of the areas A_1 and A_2 of the respective triangles:

$$\frac{CC_1}{CC_2}=\frac{A_2}{A_1}.$$

The areas are proportional to the altitudes of these triangles drawn perpendicularly to their common base. In Figure 6 the altitude a_1 is transferred to the perpendicular to C_1C_2 erected in C_2 and the altitude a_2 to the perpendicular to C_1C_2 erected in C_1. The inclined line joining the end-points of the perpendiculars intersects C_1C_2 in C. Through the law of similar triangles we get

$$\frac{CC_1}{CC_2}=\frac{a_2}{a_1}=\frac{A_2}{A_1}.$$

Therefore C is the common centroid of the triangles and the centroid of the quadrilateral.

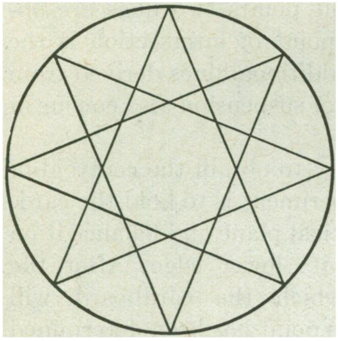

FIG. 3. Stellar octagon inscribed in a circle.

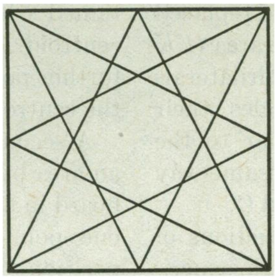

Fig. 4. Stellar octagon inscribed in a square.

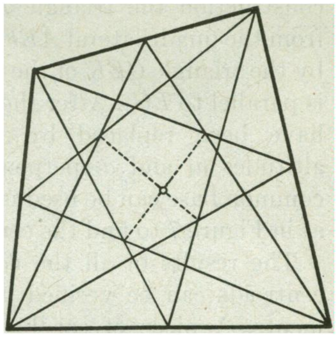

FIG. 5. Stellar octagon inscribed in a quadrilateral. Centroid.

In Figure 7 the procedure is simplified to merely transferring the distance d between C_2 and the point of intersection S of C_1C_2 with the diagonal BE: $C_2S = C_1C$. This construction is based on the fact that C_2S and C_1S are proportional to one-third of a_1 and a_2 respectively, therefore also to a_1 and a_2 themselves and to the areas of the two triangles.

These constructions can also be applied to find the centroid of polygons. The procedure is shown in Figure 8 and Figure 9 for an irregular hexagon: $ABDEFG$. In Figure 8 the diagonals from one vertex A of the hexagon are drawn: AD, AE, AF. These diagonals divide the hexagon into 4 triangles. In each triangle the centroid is obtained through the medians. From the centroids C_1 and C_2 the combined centroid C_{I} is then constructed through the method of Figure 7, and so also the combined centroid C_{II} between C_3 and C_4. Finally C_{I} and C_{II} are combined to the common centroid C. This has been carried out in Figure 9 following again the principle of Figure 7. The only difference between the constructions of Figure 9 and Figure 7 lies in the fact that Figure 7 deals with 2 triangles on a common base whose altitudes have the ratio of their areas whereas Figure 9 deals with 2 quadrilaterals. The quadrilaterals can be transformed into triangles with the common base AE: From the quadrilateral $ABDE$ the triangle BDE is cut off through the diagonal BE and is replaced by the triangle EBH which is of the same area because the line DH is drawn parallel to BE. Following the same

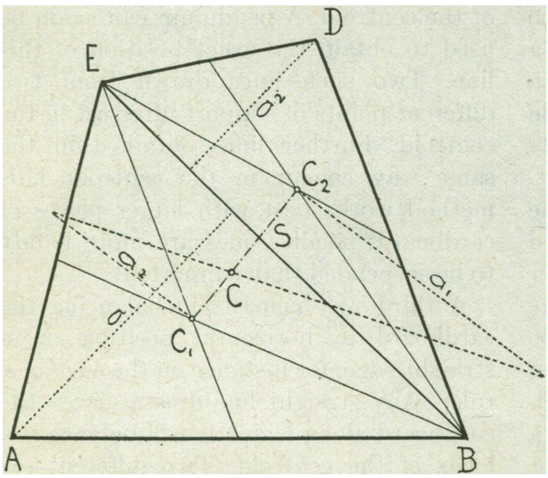

FIG. 6

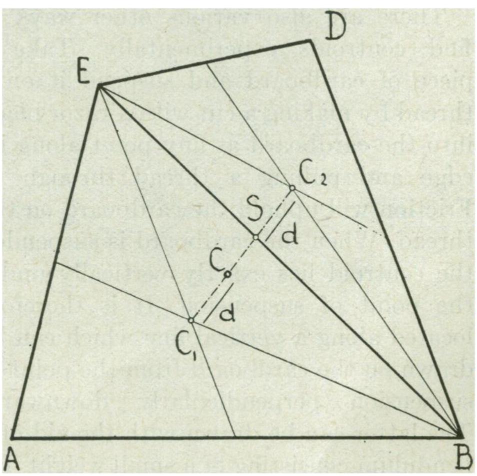

FIG. 7

Further constructions of the centroid of a quadrilateral.

construction the triangle EGF is cut off from the quadrilateral $AEFG$ and replaced by the triangle GEK of the same area (FK is parallel to EG). After the quadrilaterals have been replaced by triangles their altitudes a_1 and a_2 perpendicular to the common base can be used in the same way as in Figure 7 to find the centroid C.

The results of all the constructions of centroids can be verified through experiments. A piece of cardboard cut out in at two different points two lines are obtained whose point of intersection is the centroid. All additional lines derived from further points of suspension also concur in the centroid.

A second way to obtain the centroid of an area by experiment is to hold the cardboard in a vertical plane and balance it on one point of its lower edge. After the position in which the cardboard will balance on this point has been determined

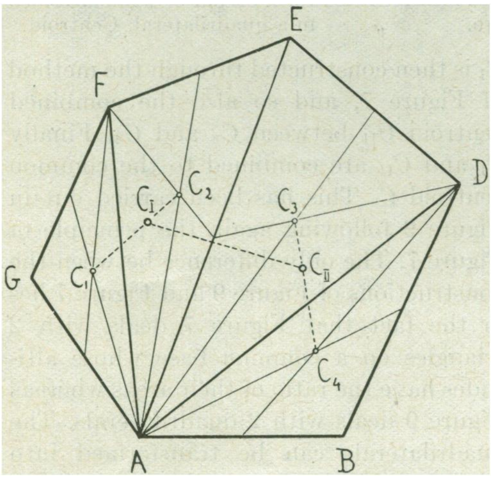

Fig. 8

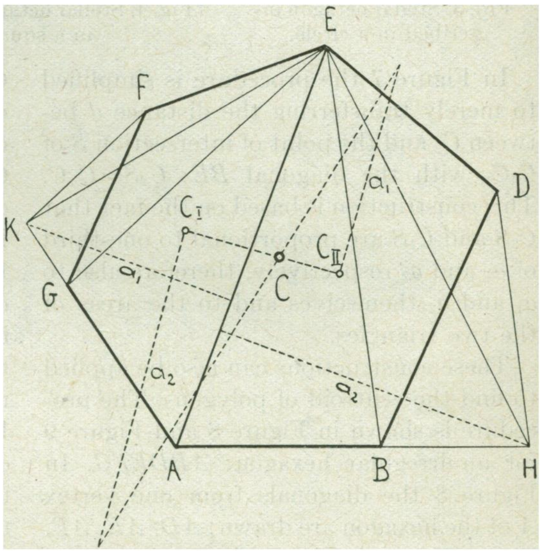

Fig. 9

Construction of the centroid of a hexagon.

shape of the given polygon will be in balance when supported at the centroid.

There are also various other ways to find centroids experimentally. Take a piece of cardboard and suspend it on a thread by making a cut with a razor blade into the cardboard at any point along its edge and pulling a thread through it. Friction will uphold the cardboard on the thread. When the cardboard is suspended the centroid lies exactly vertically under the point of suspension. It is therefore located along a vertical line which can be drawn on the cardboard from the point of suspension perpendicularly downward. The latter can be drawn with the aid of a pendulum consisting of a small weight, for instance an eraser, hanging on a thread. By suspending the cardboard successively

a vertical line drawn upward from the point of support on the cardboard is a locus of the centroid. A pendulum can again be used to obtain the exact position of this line. Two such lines drawn from two different points of support intersect in the centroid. Further lines obtained in the same way concur in the centroid. This method works best with larger pieces of cardboard; smaller ones are more handy to be suspended than supported.

A third way consists in balancing the cardboard in horizontal position on a straight edge, for instance on the edge of a ruler. Any straight line drawn across the cardboard along which it will balance is a locus of the centroid. Two different experiments yield two different straight lines and their point of intersection is the centroid. All other straight lines along which the cardboard can be balanced also concur in the centroid.

A fourth method proceeds along the way which is followed to derive the formula for the coordinates of the centroid in the calculus. It uses two rulers, the cardboard and a duplicate of the cardboard. One ruler is held in horizontal position and is balanced on the edge of the second ruler (see Figure 10). One of the cardboards is placed in horizontal position on one side of the ruler and the duplicate is hung from a thread on the other side of the ruler. By placing this thread closer to or further away from the fulcrum its position can be determined in which the two cardboards will be in equilibrium. The centroid of the horizontal piece of cardboard is located at the same distance d from the balancing edge as the thread on which the duplicate is suspended. Therefore a straight line drawn on the cardboard at a distance d from the balancing edge is a locus of the centroid. By changing the position of the cardboard which lies horizontally on the ruler and repeating the experiment a second locus for the centroid will be obtained. The point of intersection of the two loci is the centroid.

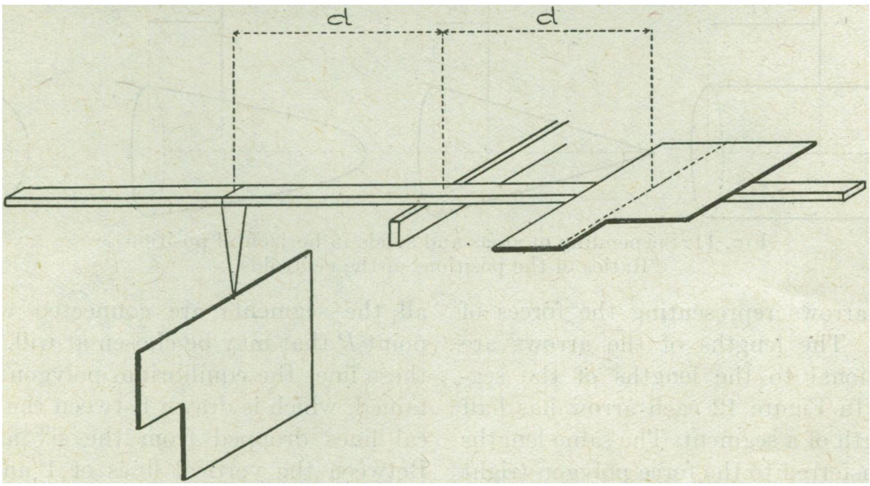

FIG. 10. Experiment to find the centroid of a cardboard-area.

An experiment which demonstrates the position of centroids for areas which have an axis of symmetry is the following: After the axis of symmetry has been marked on the cardboard one aims to suspend it on a thread in such a manner that the axis will stay in horizontal position. As every axis of symmetry is in itself a locus for the centroid the latter is its point perpendicularly under the point of suspension. The same method can be applied to solids, for instance to wooden models suspended on threads which are held to the solids by strips of scotch tape. Figure 11 shows the positions of centroids for areas and solids. The centroid divides the axis in certain ratios, as $\frac{1}{2}$ for a rectangle, prism or cylinder, $\frac{1}{3}$ for a triangle or a paraboloid, $\frac{1}{4}$ for a pyramid or a cone, $\frac{2}{5}$ for a parabola segment and $\frac{3}{8}$ for a hemisphere. For the semi-circle the constant π appears in the ratio which is $4/3\pi$. Those ratios are obtained through the calculus.

Another experiment (Figure 12) shows the balancing of a line. A wire is bent in a broken line composed of six straight line segments. The wire will stay in equilibrium with the first and last segment in horizontal position when it is suspended or supported at the point C. The figure also shows the construction of point C by the methods of graphic statics using a "force and equilibrium polygon." At the middle point of each segment are

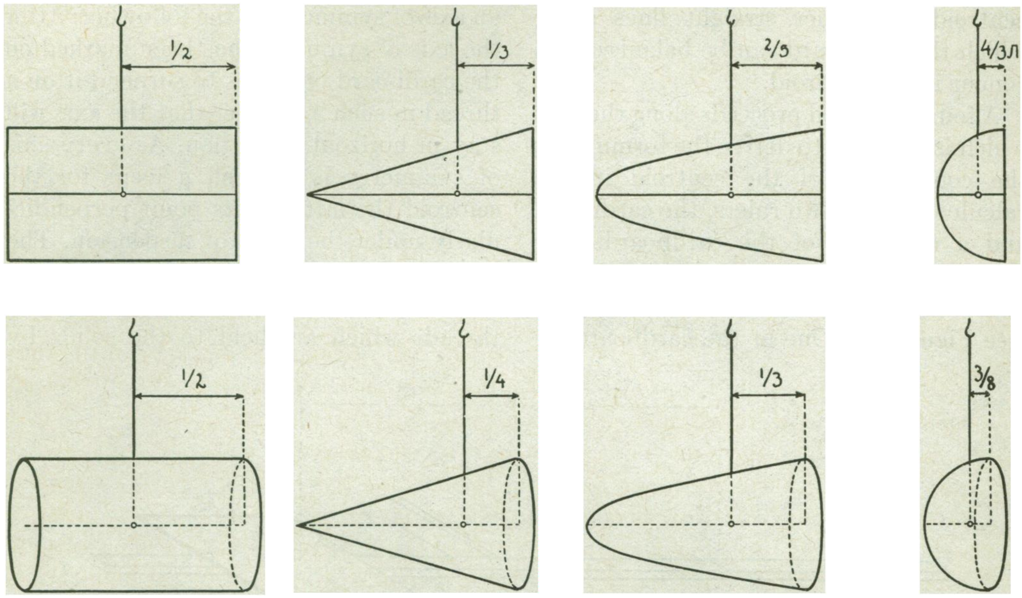

FIG. 11. Suspending of areas and solids in horizontal position.
Ratios of the positions of the centroids.

drawn arrows representing the forces of gravity. The lengths of the arrows are proportional to the lengths of the segments. In Figure 12 each arrow has half the length of a segment. The same lengths are transferred to the force polygon (right diagram of Figure 12). The end points of all the segments are connected with a point P that may be chosen at will. From these lines the equilibrium polygon is obtained, which is drawn between the vertical lines dropped from the six arrows. Between the vertical lines of 1 and 2 a parallel to the line in the force polygon

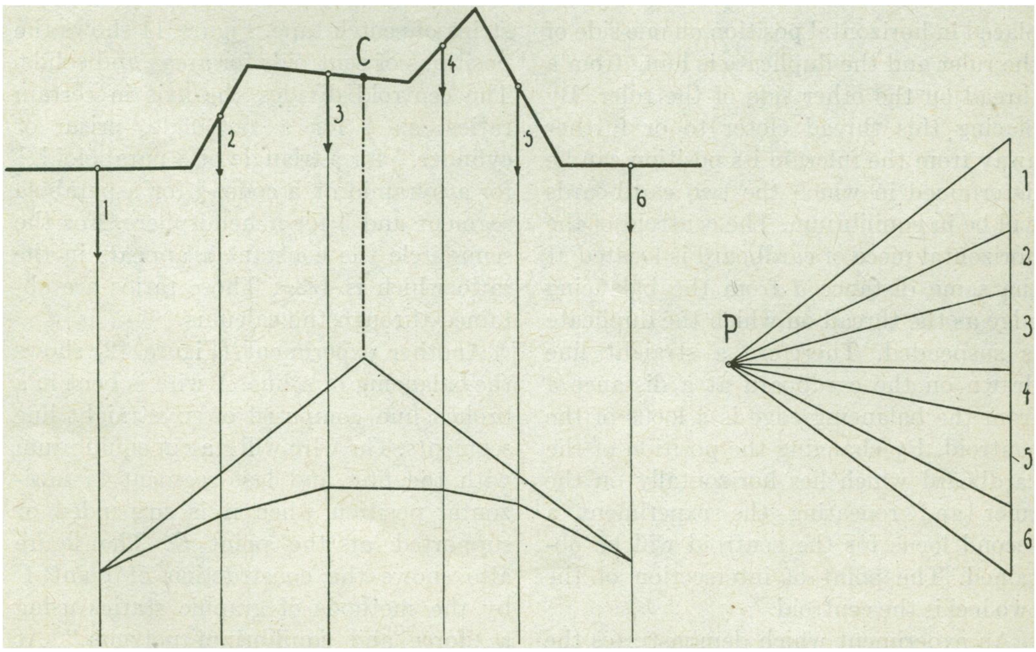

FIG. 12. Balancing of a broken line.

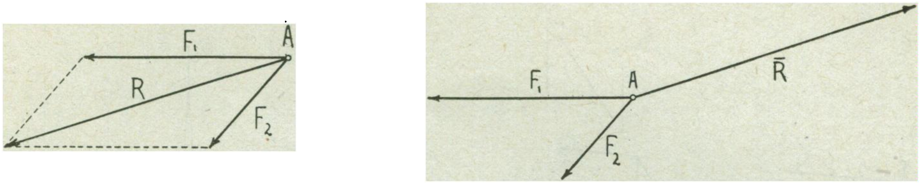

FIG. 13. Parallelogram of forces combining two concurrent forces.

connecting P with the point between the segments 1 and 2 is drawn. Then a second parallel line is drawn from the end of the first one between the verticals 2 and 3, parallel to the line connecting P with the point between the segments 2 and 3. So one continues until the last vertical is reached. Then from the end point of the last parallel a line is drawn parallel to the connection of P with the end of segment 6 and from the beginning of the equilibrium polygon a line parallel to the connecting line from P to the top of segment 1. From the point of intersection of these last two lines the vertical line which is shown in the figure in dots and dashes is drawn upwards and determines C. In this procedure three kinds of constructions are contained in a condensed form.

The first is the construction of a parallelogram of forces which combines two forces (F_I and F_2 in Figure 13) acting on an object at a point A. The resultant force R is obtained, both in its size and direction, as the diagonal of a parallelogram of forces. A force $\overline{R}$ of the same strength but acting on A in the apposite direction balances the forces F_I and F_2 and keeps the object in equilibrium (Figure 13, right diagram).

A second construction combines two forces which act on two different points (Force F_1 acting on point A and Force F_2 on point B which both lie on a rectangular board drawn in Figure 14). The construction proceeds by extending the arrows of the given forces backwards until they intersect. At the point of intersection C the parallelogram of forces is drawn which yields the resultant R. A force $\overline{R}$ equal in strength to the resultant R but acting in the opposite direction on any point C along the diagonal of the parallelogram of forces will keep the board in equilibrium (Figure 14, right diagram).

The construction of Figure 14 can be carried out for any two forces except for parallel forces. But cases of parallel forces occur especially frequently as they include the forces of gravity. Parallel forces can be handled through the trick of adding an arbitrarily chosen pair of equal opposite forces. The procedure is shown in Figure 15, left diagram. The given forces are F_1 acting on point A and F_2 acting on point B. The additional pair of forces consists of G_1 and G_2. First F_1 and G_1 are combined in a parallelogram of forces and their resultant is R_1. Then R_1 and F_2 are combined as non-concurrent forces and their resultant is R_2. Finally R_2 and G_2 are combined as non-concurrent forces and their resultant R is directed perpendicularly downwards, its length being equal to the sum of the lengths of the arrows F_1 and F_2.

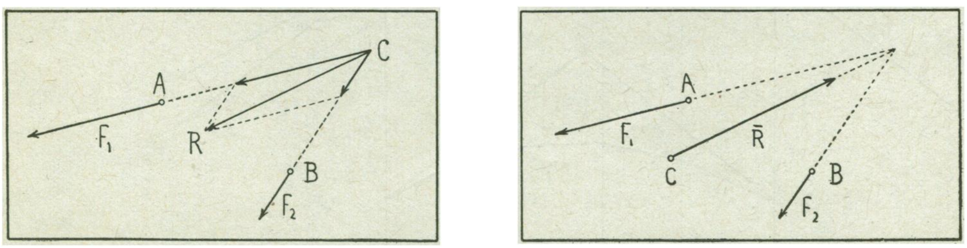

FIG. 14. Parallelogram of forces for two non-concurrent forces.

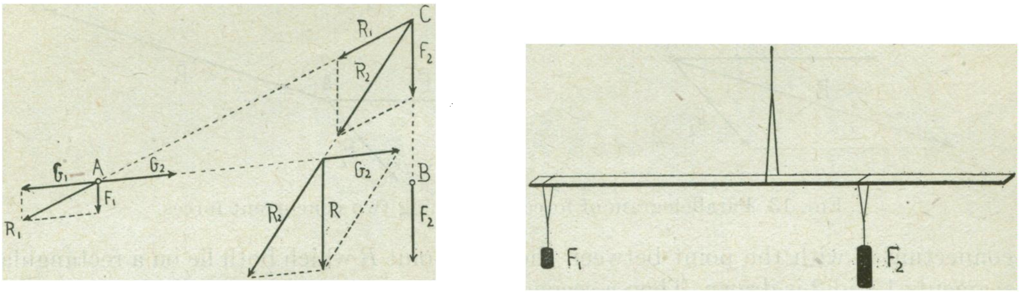

FIG. 15. Resultant of two parallel forces.

A force $\overline{R}$ of the same strength as R but with the opposite direction acting on the same point as R restores the equilibrium. In the right diagram the forces F_1 and F_2 are interpreted as weights suspended on a ruler. The ruler stays in equilibrium when supported at a point with the same distances from the given forces as R.

Repeated application of this construction solves the problem to find the centroid of the wire in Figures 12 and 16. The forces of gravity which apply to the 6 sections of the wire are denoted simply as 1, 2, 3, 4, 5 and 6. The additional pair of forces is G_1 and G_2. First 1 and G_1 are combined in a parallelogram of forces which yields a resultant. Then this resultant is further combined with force 2 and so forth until one arrives at the last resultant which is finally combined with G_2. Thus the position of the ultimate resultant is reached. On the right hand of this diagram the force polygon of Figure 12 is repeated. In it the lengths of the segments 1 to 6 equal those of the arrows 1 to 6. The position of the point P has been so chosen that PA equals G_1. A systematic comparison of the force polygon with the diagram on the left side of Figure 16 will show that the force polygon is but a condensation of this

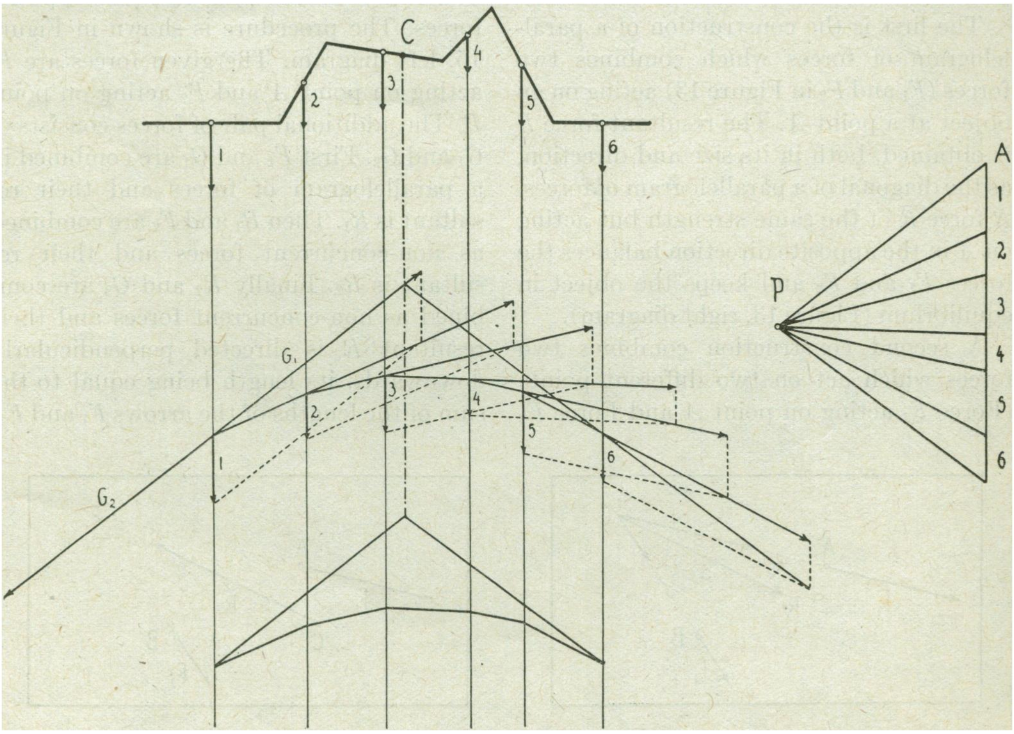

FIG. 16. Derivation of the construction of the centroid by force and equilibrium polygons.

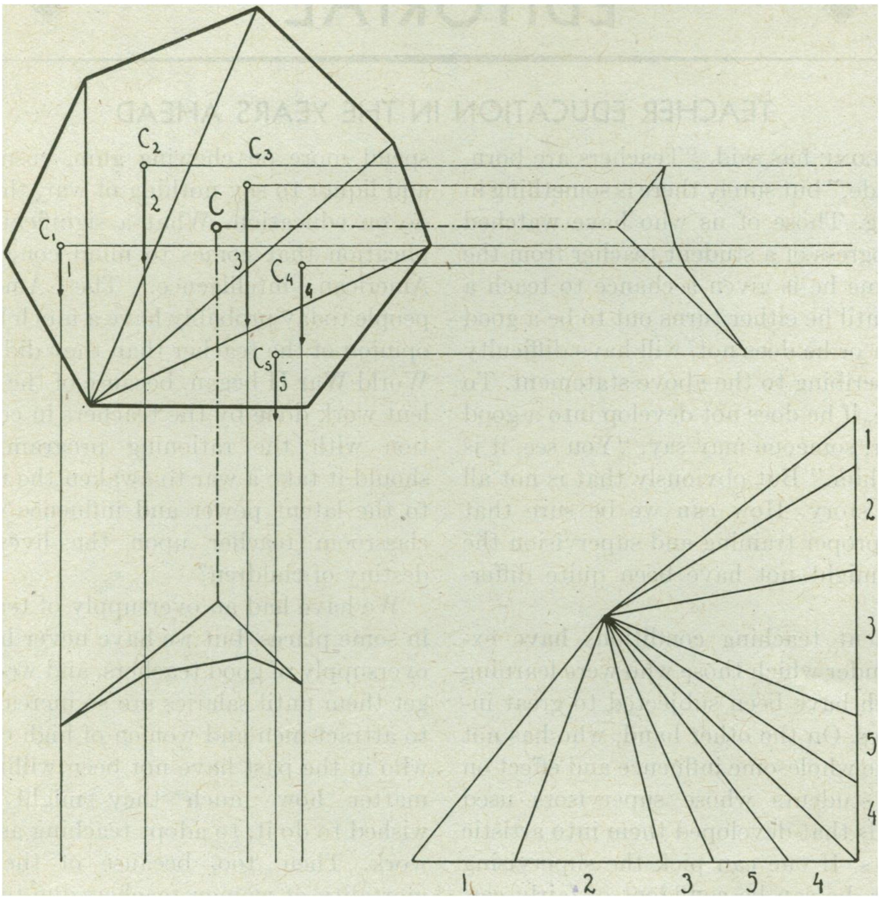

FIG. 17. Construction of the centroid of an area through force and equilibrium polygons.

diagram and is composed of triangles which are congruent to those which appear in the parallelograms of forces. Thus the construction of the force and equilibrium polygons of Figures 12 and 16 derives itself from the basic constructions of the Figures 13 to 15.

Finally the construction of force and equilibrium polygons can be applied to an irregular polygon, as in Figure 17. The diagonals drawn from one vertex of the polygon divide it into triangles. For each triangle the centroid has been obtained (through medians not shown in the diagram). From the centroids of the triangles arrows are drawn representing the forces of gravity whose lengths are proportional to the areas of the triangles. From the lengths of these arrows the force polygon has been set up and through it the equilibrium polygon which is a locus of the centroid (See dots and dashes). Imagining then that the given area with the diagram be turned 90° so that the parallel lines which are drawn from the centroids of the triangles to the right come into perpendicular position, the construction of the force and equilibrium polygon can be repeated once more. Thus a second line shown in dots and dashes is obtained which is also a locus for the centroid of the area. The point of intersection of the 2 loci is the final centroid C. The same construction can be carried out without an excessive amount of lines to find the centroid of any given polygon. Its results will be the same whatever vertex may have been chosen to draw the diagonals across the polygon or whatever decision has been made to divide the area into triangles. The centroid thus obtained can be checked by experiment.