Some Circle Concurrency Theorems

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INTRODUCTION

The random drawing or scribbling of squiggles, diagrams, figures or patterns on paper by someone is often called 'doodling'. While doodling can be seen as a form of daydreaming, and often done to relax and while away the time and/or to combat boredom (such as in a lengthy meeting), it can sometimes produce new, creative ideas. Recently I was idly making some random geometric constructions using dynamic geometry software – for me, the geometric version of doodling – in the hope of perhaps seeing something interesting come up. While most of my constructions led nowhere, I suddenly found the following interesting result.

THEOREM 1

> "Given any ΔABC with triangles ADB, BFC and CEA constructed on its sides such that $\angle D + \angle E + \angle F = 360°$, then the three circumcircles of ADB, BFC and CEA are concurrent (at say P)" (see Figure 1).

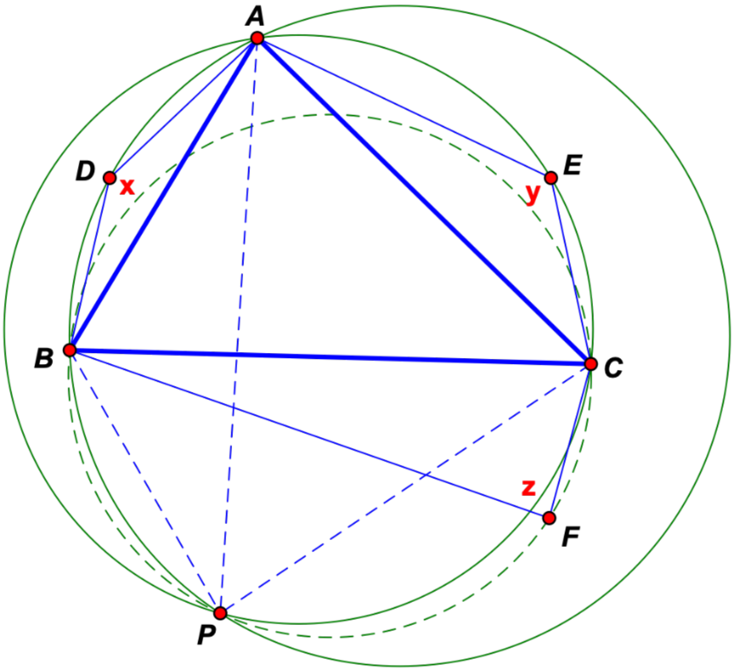

FIGURE 1: Circumcircles of triangles ADB, BFC and CEA concurrent at P.

The reader is invited to view and manipulate a dynamic version of this result (and the others further on) at http://dynamicmathematicslearning.com/circle-concurrencies.html and also to attempt proving it before reading further. The results are not hard to prove and should be readily accessible for Grade 11-12 learners acquainted with circle geometry.

PROOF

To prove the result, assume that the circumcircles of ADB and CEA meet at P. We now need to prove that the circumcircle of BFC also passes through P. But this is equivalent to proving that $BPFC$ is a cyclic quadrilateral.

Let the angles at D, E and F respectively be x, y and z as shown in Figure 1. Now note that since $ADBP$ and $AECP$ are cyclic, it respectively follows that $\angle APB = 180° - x$ and $\angle APC = 180° - y$ (opposite angles of a cyclic quadrilateral). Thus, $\angle BPC = 360° - x - y$, but it is given that $x + y + z = 360°$. Hence, by substitution, $\angle BPC = z = \angle BFC,$ which implies that $BPFC$ is cyclic, since we have shown that equal angles are subtended on chord BC. This then completes the proof.

Note that Theorem 1 can also be formulated as follows: "Given any hexagon $ADBFCE$ with the sum of alternate angles $\angle D + \angle E + \angle F = 360°$, then the three circumcircles of ADB, BFC and CEA are concurrent." Clearly, when the hexagon $ADBFCE$ is regular, or more generally is cyclic, the three circumcircles of these triangles will coincide. In addition, it is not hard to show from the above condition that the circumcircles of EAD, DBF and FCE are also concurrent (in a different point), and this is left to the reader as an exercise. (In the dynamic sketch in the URL given earlier, click on the "*Show Objects*" button to view the concurrency at P' of these three circles).

SOME APPLICATIONS

This elementary theorem can be applied in several situations. For example, if on the sides of any ΔABC triangles AXB, $BYC$and CZA are constructed as shown in Figure 2 so that $\angle X + \angle Y + \angle Z = 180°$, and the respective incentres D, E and F of triangles AXB, BYC and CZA are constructed, then the circumcircles of triangles ADB, BEC and CFA are also concurrent at a point P.

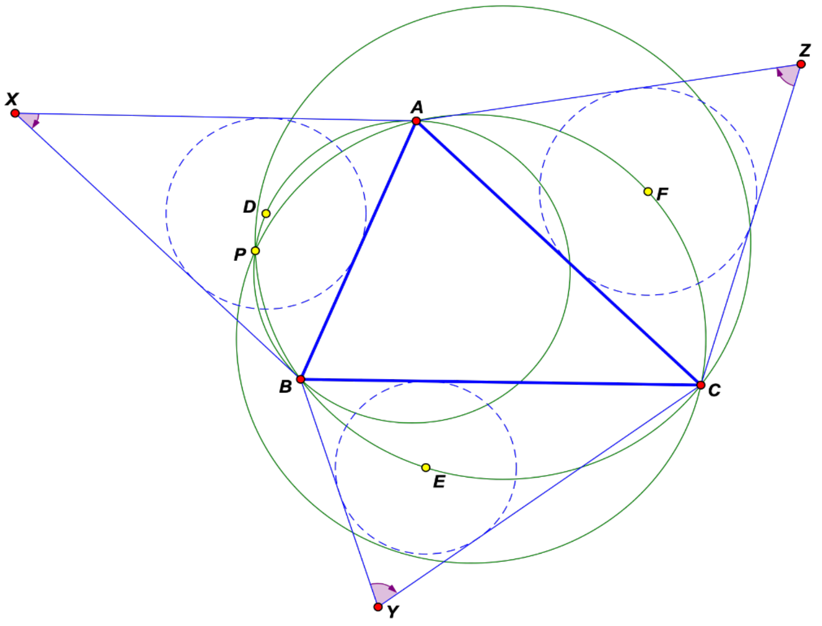

FIGURE 2

The proof is quite simple and follows from the well-known property of an incentre that $\angle ADB = 90° + \angle X/2$, $\angle BEC = 90° + \angle Y/2$ and $\angle CFA = 90° + \angle Z/2$. From this it follows that:

$$\angle D + \angle E + \angle F = 270° + (\angle X + \angle Y + \angle Z)/2 = 270° + 90° = 360°$$

It therefore meets the condition of the theorem, and the result follows.

Here is another example. If for the same configuration given in Figure 2, the circumcentres D, E and F of triangles AXB, BYC and CZA are constructed, then the circumcircles of triangles ADB, BEC and CFA are also concurrent at another point P. It is easy to see why this is so since the angle subtended at the circumcentre of each triangle is twice the one on the circumference. Hence, $\angle D + \angle E + \angle F = 2\angle X + 2\angle Y + 2\angle Z = 360°$, and again meets the condition of Theorem 1.

Next we shall present two famous theorems about circle concurrencies (and triangle similarity) that deserve to be better known by high school teachers & learners.

A GENERALISATION OF NAPOLEON'S THEOREM

The celebrated Napoleon's theorem states that the circumcentres of equilateral triangles constructed (outwardly or inwardly) on the sides of any triangle produce another equilateral triangle. While the theorem is named after Napoleon Bonaparte (1769-1821), the famous French Emperor, it is historically doubtful whether he actually discovered or proved the theorem (see Grünbaum, 2012).

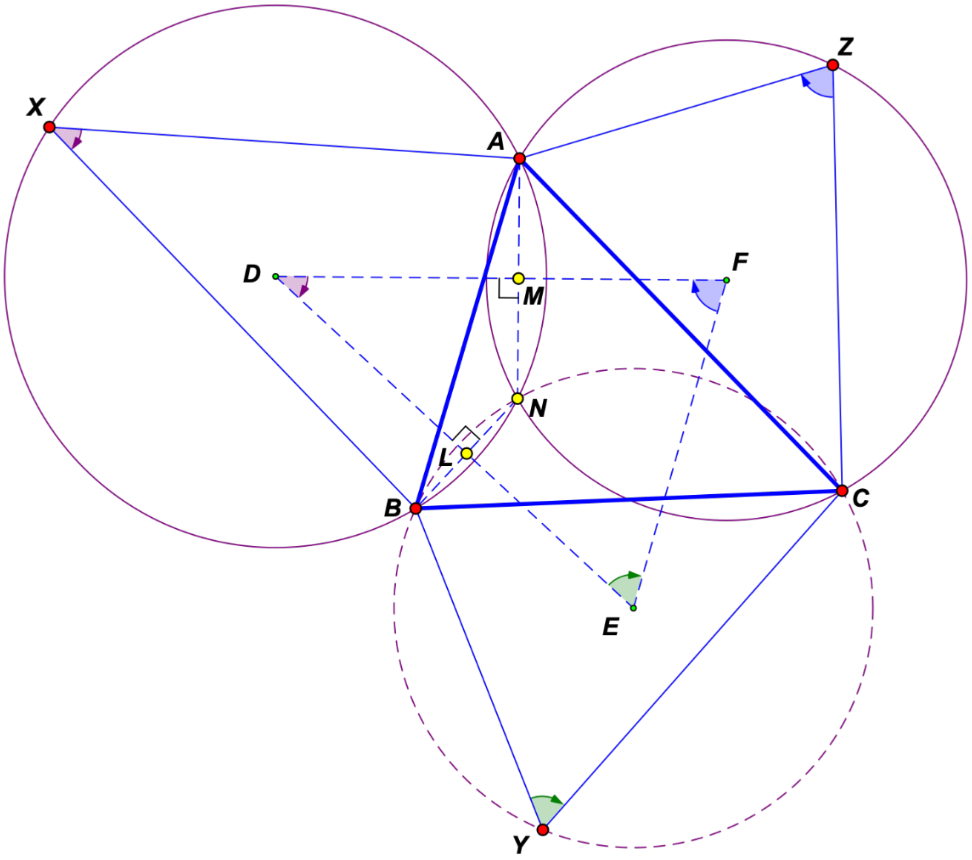

FIGURE 3

Napoleon's theorem can be neatly generalised as follows (Coxeter & Greitzer, 1967). If for the same configuration given in Figure 2, the circumcircles and circumcentres D, E and F of triangles AXB, BYC and CZA are constructed, then the circumcircles of triangles AXB, BYC and CZA are concurrent at a point N, and $\angle FDE = \angle X$, $\angle DEF = \angle Y$ and $\angle EFD = \angle Z$ (see Figure 3).

PROOF

Assume that the circumcircles of AXB and CZA meet at N. We now need to prove that $BYCN$ is cyclic. But from the assumption we have $\angle ANB = 180° - \angle X$ and $\angle ANC = 180° - \angle Z$. Thus $\angle BNC = 360° - (180° - \angle X) - (180° - \angle Z) = \angle X + \angle Z = 180° - \angle Y$ (since it is given that $\angle X + \angle Y + \angle Z = 180°$). Since $\angle BNC$ is supplementary to $\angle Y$ it follows that $BYCN$ is cyclic, and that the three circumcircles are concurrent.

Now note from the equal radii of the circles that $DAFN$ and $DBEN$ are kites. Since the diagonals of kites are perpendicular, we have $\angle DMN = 90° = \angle DLN$. Therefore, $DMNL$ is cyclic (opposite angles supplementary). Thus, $\angle FDE = 180° - \angle ANB = 180° - (180° - \angle X) = \angle X$. Similarly it can be shown that $\angle DEF = \angle Y$ and $\angle EFD = \angle Z$.

MIGUEL'S THEOREM

The following theorem was originally discovered by Auguste Miquel in 1838, and is sometimes also called the "Pivot" theorem. It can be formulated as follows: If three arbitrary points A, B and C are respectively chosen on the sides XZ, XY and YZ of any ΔXYZ, then the circumcircles of triangles AXB, BYC and CZA are concurrent at a point N, and the triangle DEF formed by the respective circumcentres is similar to ΔXYZ (see Figure 4).

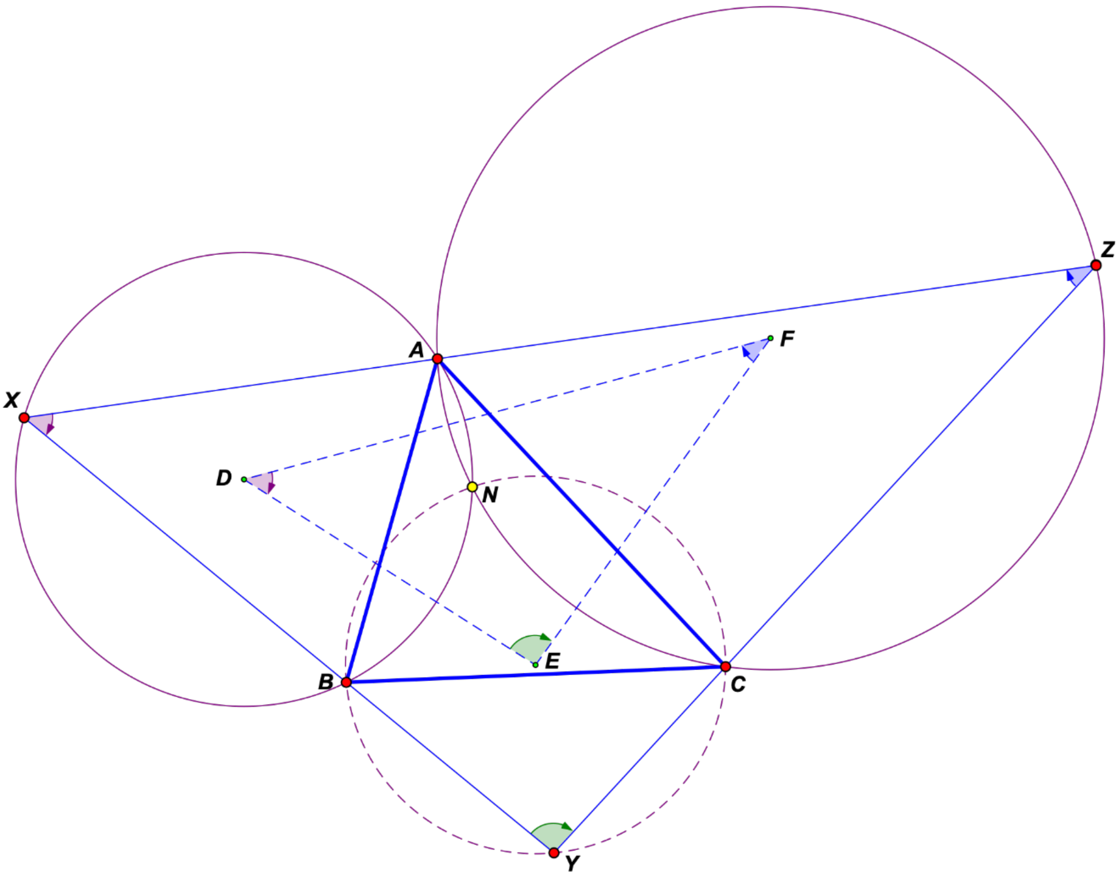

FIGURE 4

While Miquel's theorem can be proved independently from the preceding generalization of Napoleon's theorem, it is elegant and easy to simply view it as a special case of this generalization. With reference to Figure 2, the special case occurs when XAZ, XBY and YCZ all lie in straight lines i.e. are collinear. Or stated differently, Miquel's theorem is immediately obtained from the earlier Napoleon generalization when ΔABC lies on the sides of ΔXYZ.

Some further interesting questions to possibly explore with mathematically talented learners in relation to the preceding results are to consider their converses, as well as possible generalisations to higher polygons (e.g. see De Villiers, 2014; Wetzel, 1992).

CONCLUDING COMMENTS

Unfortunately South African learners are no longer required to know about the concurrency of the medians, angle bisectors, altitudes and perpendicular bisectors of a triangle. This is a pity as these concurrency proofs provide a good introduction to logical thinking and the verification and explanation of non-obvious results. For example, Albert Einstein remarked in his autobiography about the indelible impression the concurrency of the altitudes, and its proof, made on his young mind and inspired him in the direction of the mathematical sciences (see Pyenson, 1985).

In this short paper some circle concurrency theorems have been discussed that should be easily accessible for Grade 11-12 South African learners. Together with the use of dynamic geometry software for initial exploration as utilized in De Villiers (1999; 2003; 2012), the proving of such concurrency results can substantially enrich the current geometry curriculum, and help provide meaningful intellectual challenges for our more mathematically talented learners.

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