A Surprise Equilateral Triangle

Michael de Villiers

RUMEUS, University of Stellenbosch

profmd1@mweb.co.za

Introduction

Recently, while working on another problem, I came across the following surprising equilateral triangle result:

> Given any hexagon $ADBFCE$ with $\angle D = \angle E = \angle F = 120°$ as shown in Figure 1, then the respective circumcentres D', F' and E' of the three circumcircles of triangles ADB, BFC and CEA form an equilateral triangle.

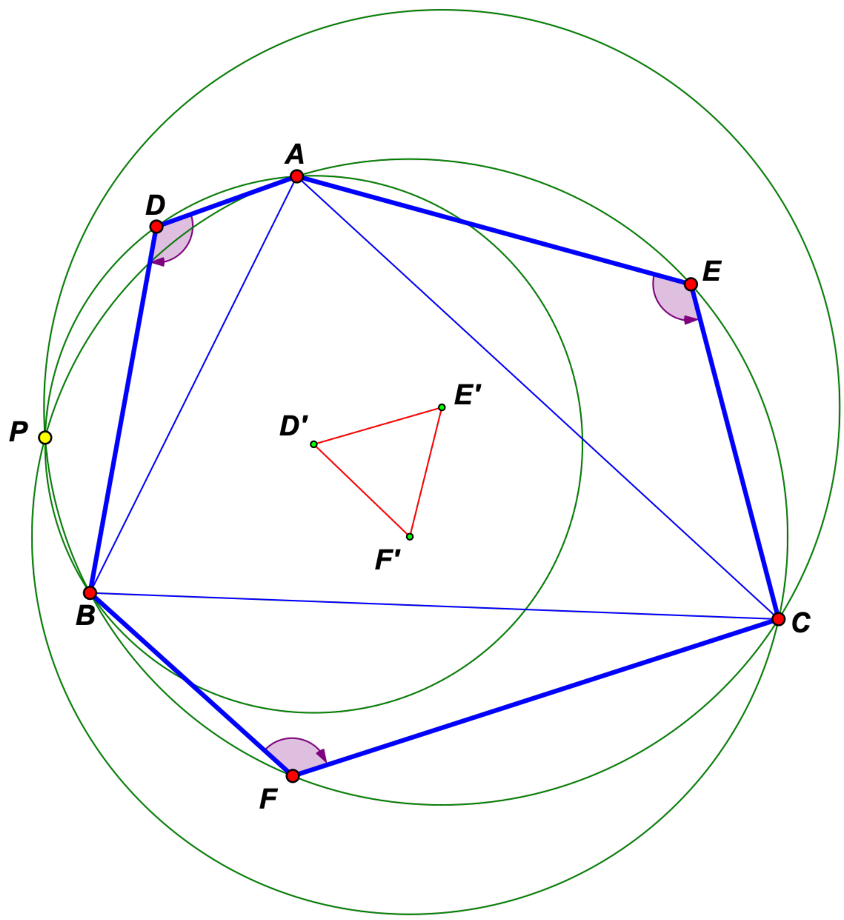

Figure 1: The formed equilateral triangle

Note that the result can also be formulated in the following equivalent form:

> Given any triangle ABC with triangles ADB, BFC and CEA constructed on its sides so that $\angle D = \angle E = \angle F = 120°$, then the respective circumcentres D', F' and E' of the three circumcircles of triangles ADB, BFC and CEA form an equilateral triangle.

The reader may wish to check the result using dynamic geometry for themselves, but a dynamic geometry sketch is available online for use at: https://dynamicmathematicslearning.com/circle-concurrencies.html (navigate to the correct sketch by clicking on the 'Link to Special Napoleon Variation' button).

While dragging and moving a figure around in dynamic geometry is a very useful and convincing way to check the validity of results, it does not explain *why* the result is true. Before continuing the reader is encouraged first to try and prove the result themselves.

The astute reader may already have recognized that the result is directly related to Napoleon's theorem discussed in my paper in *Learning and Teaching Mathematics*, No. 33 (De Villiers, 2022). In that paper a generalisation of Napoleon's theorem to some more general triangles is given, but for the purpose of proving the above result we only need the original form of Napoleon's theorem, namely: "If equilateral triangles are constructed on the sides of any ΔABC, then the centroids of the equilateral triangles form an equilateral triangle".

Note that, as shown in Figure 2, the equilateral triangles can be constructed to the exterior as well as to the interior of the base triangle. Obviously, the respective centroids of these equilateral triangles coincide with their respective circumcentres. Many proofs for this famous theorem are available in books as well as on the internet. A free classroom worksheet with related downloadable Sketchpad sketches[1] is also available in De Villiers (2012). The proof of the case when the equilateral triangles are drawn (or dragged in a dynamic sketch) towards the interior is a little more difficult to visualise but is exactly the same.

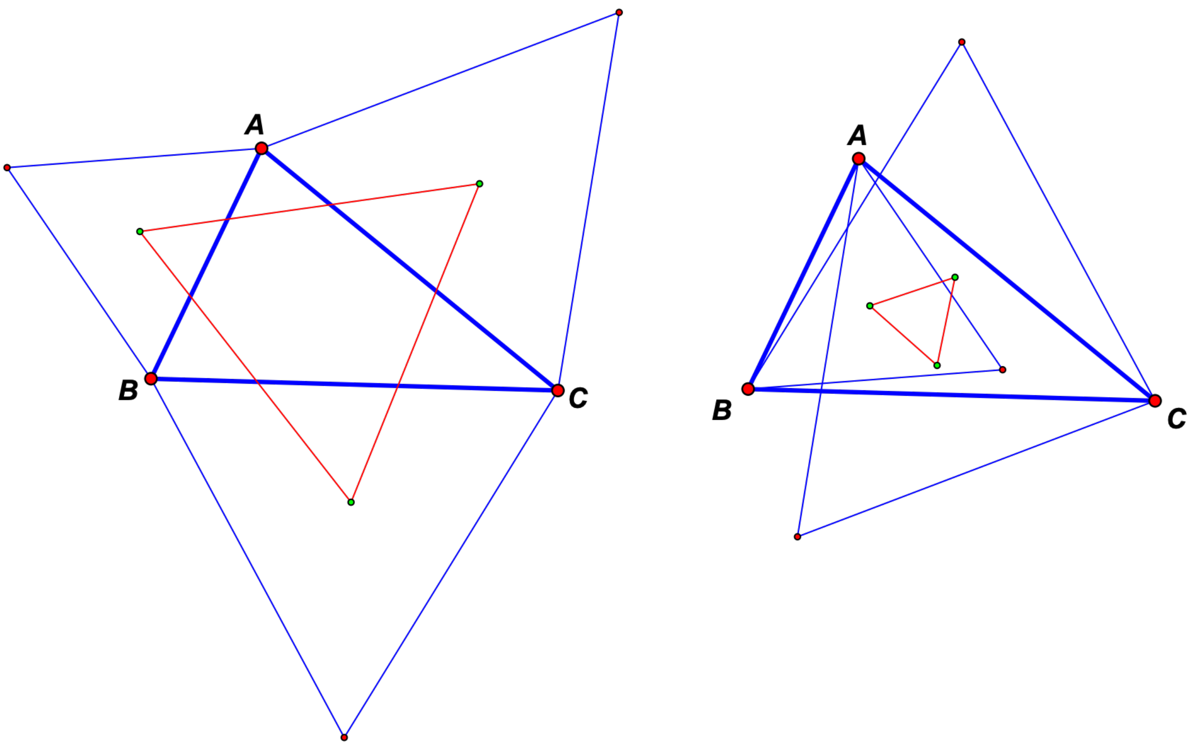

FIGURE 2: Napoleon's Theorem

[1] *Sketchpad* is now free to download at: http://dynamicmathematicslearning.com/free-download-sketchpad.html

Proof of the surprise result

We are now ready to prove the surprise result at the start. Consider again Figure 1 and draw an equilateral triangle inwardly on side AB, as shown in Figure 3, and construct its circumcircle. Since $\angle D = 120°$ and $\angle AXB = 60°$, it follows that X lies on the circumcircle of ΔADB. Therefore, its circumcentre D' coincides with that of the circumcentre of equilateral ΔAXB. If we repeat the same construction on the other sides, it should now be clear that we simply obtain the same Napoleon equilateral triangle as in the second case in Figure 2. This completes the proof.

In essence the surprise result turns out to be merely a simple variation of the standard theorem of Napoleon with the equilateral triangles constructed inwardly. Obviously if we construct triangles ADB, BFC and CEA inwardly, we will merely obtain the outward Napoleon equilateral triangle.

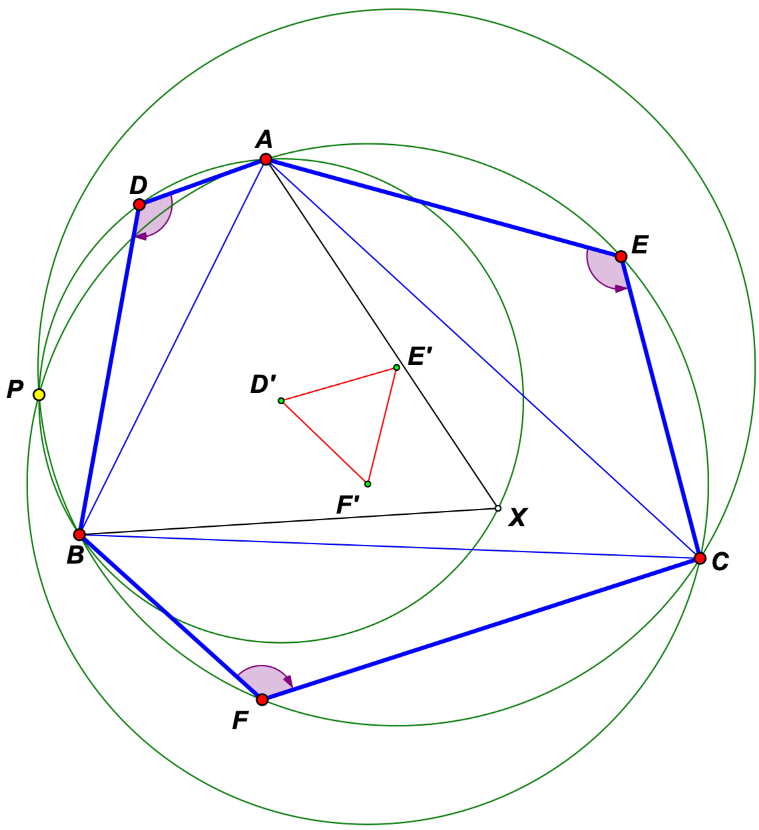

Figure 3: Proof of the surprise equilateral triangle result

References

De Villiers, M. (2012). In *Rethinking Proof with Geometer's Sketchpad*, pp. 119-121. Available at: https://www.researchgate.net/publication/375342639_Rethinking_Proof_with_Geometer%27s_Sketchpad

De Villiers, M. (2022). Some circle concurrency theorems. *Learning and Teaching Mathematics*, No. 33, pp. 34-38.