A tour around Quadrilateral Geometry / Darij Grinberg

In this note we will perform a little mental tour around Elementary Geometry. We begin with a property of general quadrilaterals; specifying on circumscribed quadrilaterals, we will establish a property of them, and finally pass over back to general quadrilaterals by proving the converse of this property. The results I am going to deal with are not new themselves; however, their interconnection shown in this note seems to be as new as 2004, discovered by Marcello Tarquini [3].

§1. The four excircles of a quadrilateral

(See Fig. 1.) Consider an arbitrary quadrilateral $ABCD$. The external angle bisectors of the angles DAB, ABC, BCD, CDA will be abbreviated as *external angle bisectors of* A, B, C, D.

The external angle bisectors of A and B meet at X. The external angle bisectors of B and C meet at Y. The external angle bisectors of C and D meet at Z. The external angle bisectors of D and A meet at W.

If F is the point of intersection of BC and DA, then (in the configuration of Fig. 1) the point X is the meet of the angle bisectors of FAB and ABF, i. e. the incenter of triangle FAB. Thus, X is the center of a circle touching the segment AB and the segments FA and FB (i. e., the extensions of the segments BC and DA). Analogous statements are valid for Y, Z, W. Altogether, the points X, Y, Z, W are the centers of four circles, each of these touching one side of the quadrilateral $ABCD$ internally and the extensions of the two adjacent sides. These four circles will be called *excircles* of the quadrilateral $ABCD$.

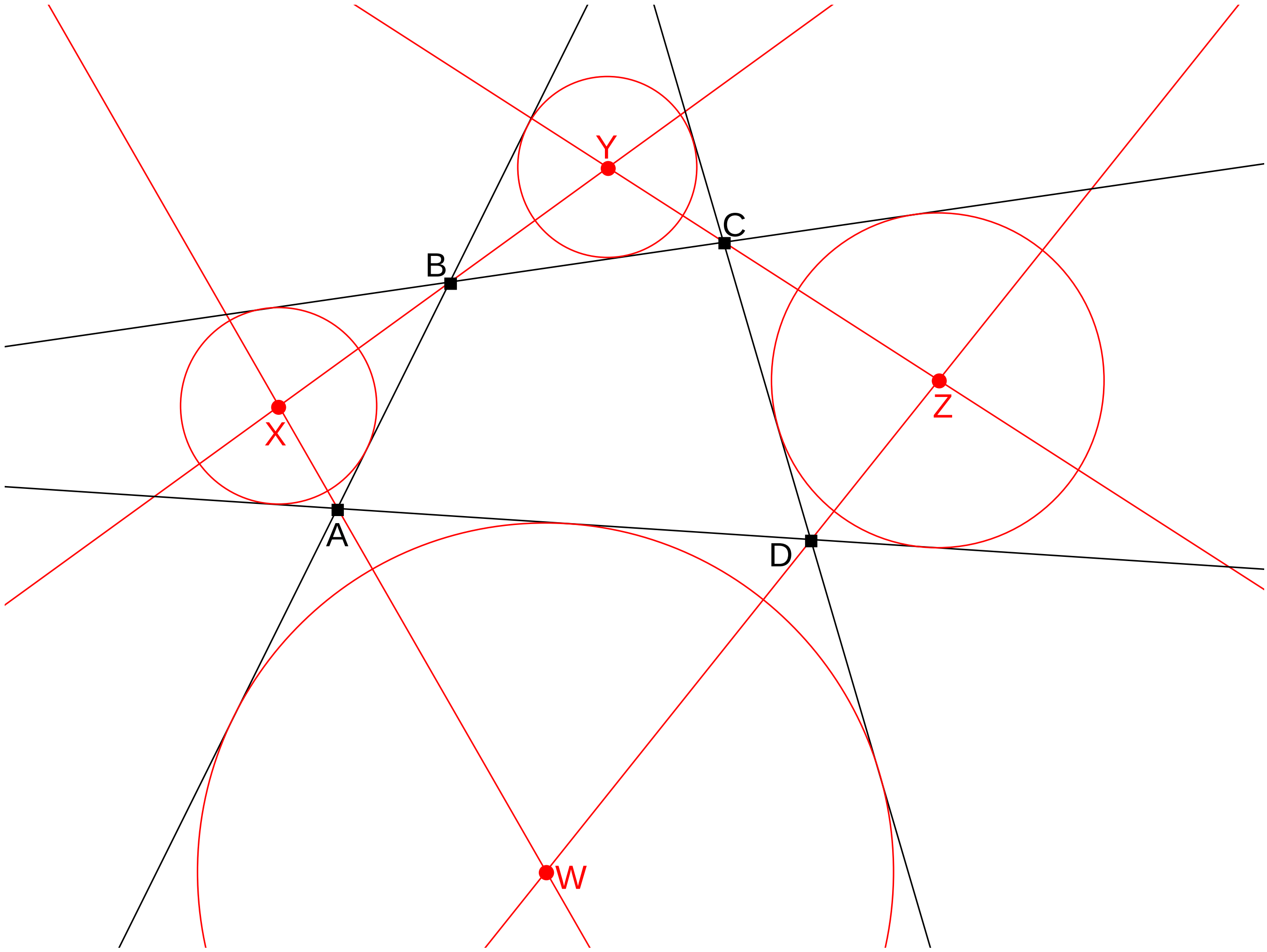

Fig. 1

We follow my note [4] and show a theorem dating back to Hadamard [2, Problem 66], if not already known earlier:

Theorem 1. The points X, Y, Z, W lie on a circle, i. e. the quadrilateral $XYZW$ is inscribed (Fig.

2).

The *proof* is not spectacular. Since Z lies on the external angle bisectors of BCD and CDA, we have $\measuredangle ZCD = 90° - \frac{1}{2}\measuredangle BCD$ and $\measuredangle ZDC = 90° - \frac{1}{2}\measuredangle CDA$, hence

$$\measuredangle YZW = \measuredangle CZD = 180° - \measuredangle ZCD - \measuredangle ZDC$$
$$= 180° - \left(90° - \frac{1}{2}\measuredangle BCD\right) - \left(90° - \frac{1}{2}\measuredangle CDA\right) = \frac{1}{2}\measuredangle BCD + \frac{1}{2}\measuredangle CDA.$$

Similarly,

$$\measuredangle WXY = \frac{1}{2}\measuredangle DAB + \frac{1}{2}\measuredangle ABC.$$

This entails

$$\measuredangle YZW + \measuredangle WXY = \frac{1}{2}\measuredangle BCD + \frac{1}{2}\measuredangle CDA + \frac{1}{2}\measuredangle DAB + \frac{1}{2}\measuredangle ABC$$
$$= \frac{1}{2}(\measuredangle BCD + \measuredangle CDA + \measuredangle DAB + \measuredangle ABC) = \frac{1}{2} \bullet 360° = 180°.$$

Hence, the quadrilateral $XYZW$ is inscribed, what proves Theorem 1.

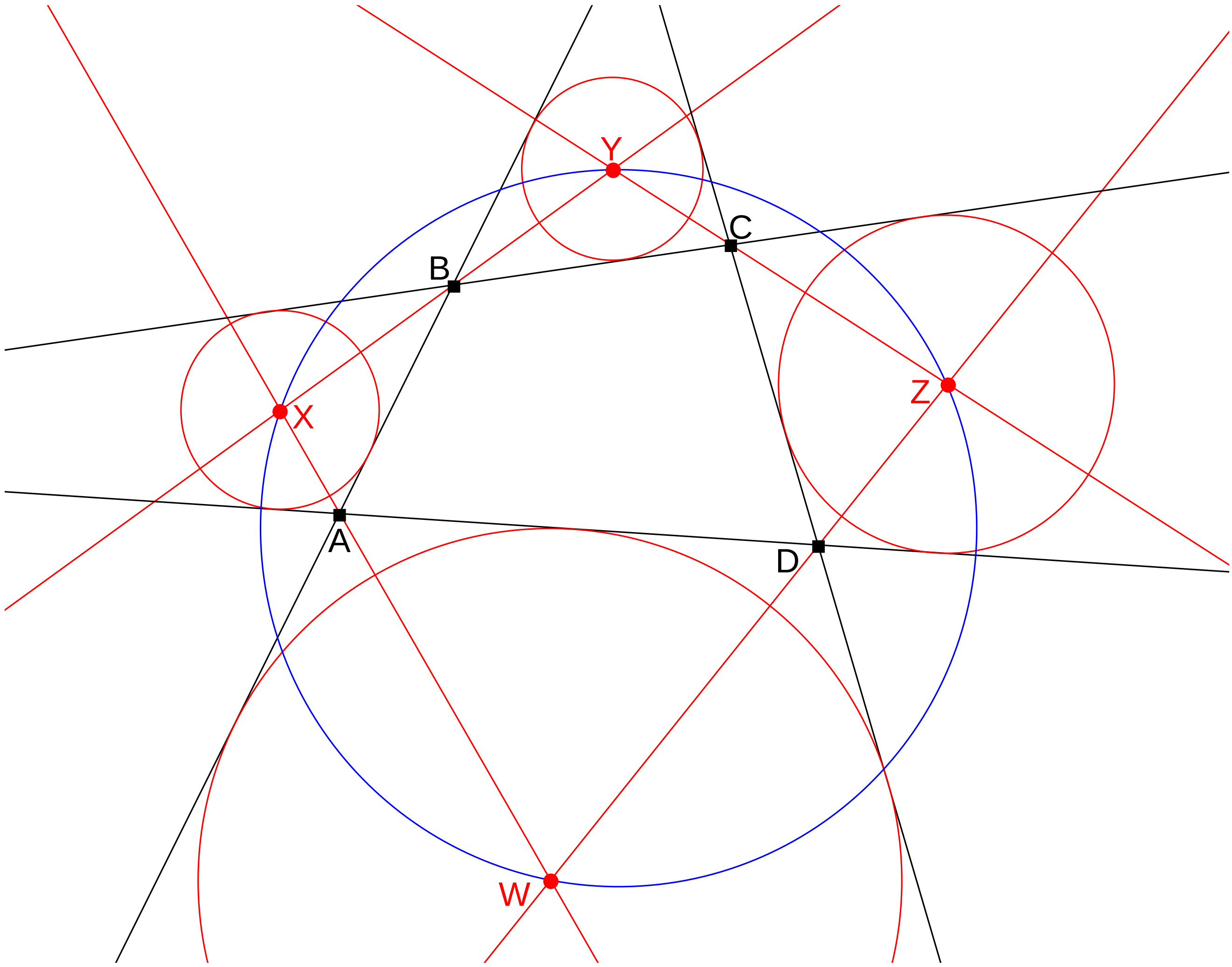

Fig. 2

Theorem 1 is not very extraordinary in itself, but it is fundamental for the observations we will make afterwards. Note that Theorem 1 is contained in Satz 1.10.1 of [1].

§2. The circumcenter of *XYZW*

One of my own results shown in [4] is:

Theorem 2. The perpendiculars to AB, BC, CD, DA through the points X, Y, Z, W (i. e., the radii of the excircles to their internal tangency points with the sides) are equidistant from the circumcenter M of the quadrilateral $XYZW$. In other words, there exists a circle centered at M and touching the perpendiculars to AB, BC, CD, DA through X, Y, Z, W (Fig. 3).

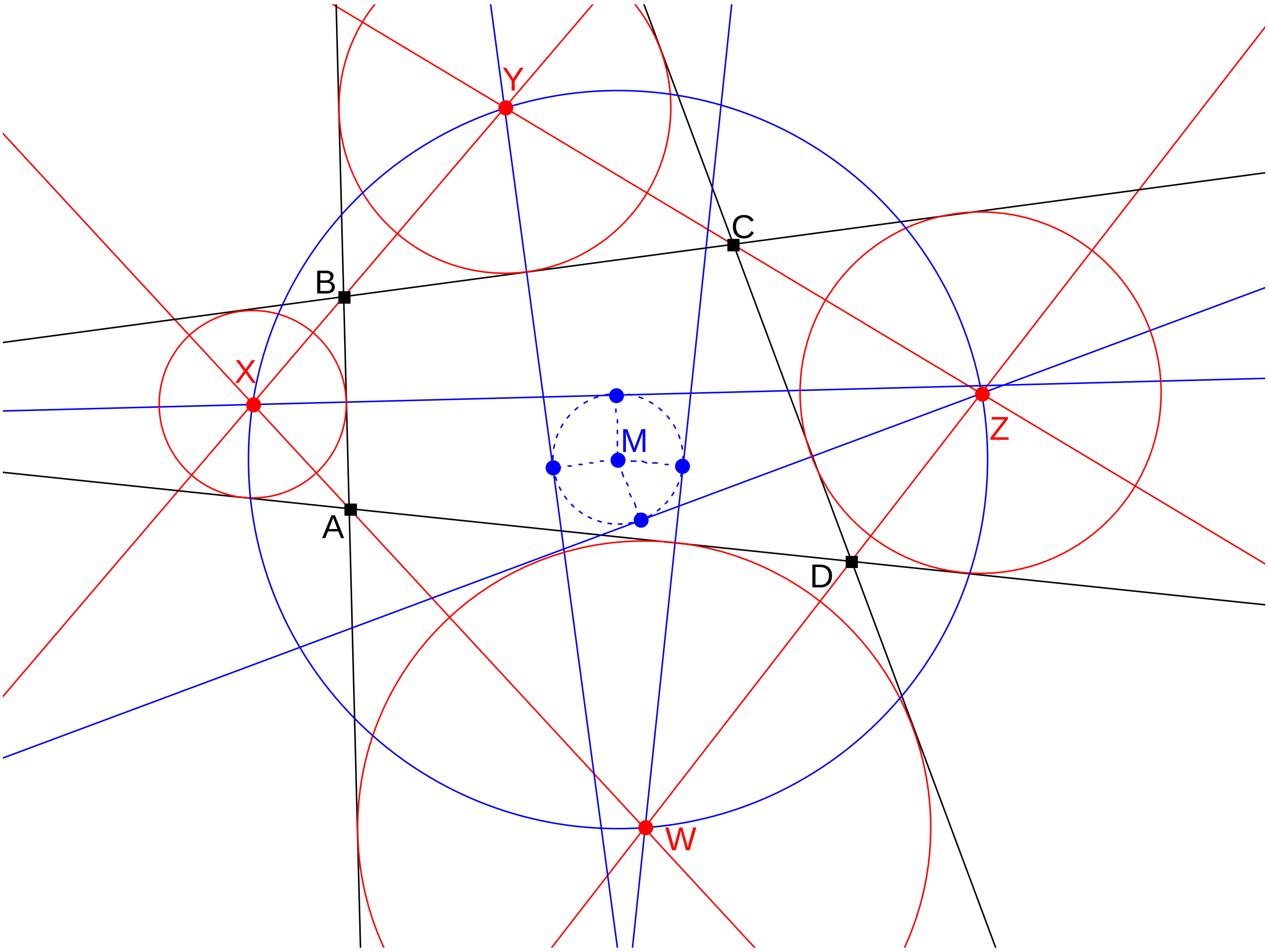

Fig. 3

The following *proof* (Fig. 4) is a simplification of the proof I gave in [4]: Let the perpendiculars to AB and BC through X and Y meet the lines AB and BC at X' and Y', and let X'' and Y'' be the projections of M on these two perpendiculars.

As the line XY is the external angle bisector of ABC, we have $\measuredangle XBX' = \measuredangle YBY'$. In the right-angled triangles $BX'X$ and $BY'Y$, we have $\measuredangle BXX' = 90° - \measuredangle XBX'$ and $\measuredangle BYY' = 90° - \measuredangle YBY'$; thus, $\measuredangle BXX' = \measuredangle BYY'$, i. e. $\measuredangle YXX'' = \measuredangle XYY''$. On the other hand, $YM = XM$, for M is the center of the circle through X, Y, Z, W, and consequently, ΔXMY is isosceles, and $\measuredangle YXM = \measuredangle XYM$. Therefore, $\measuredangle Y''YM = \measuredangle XYM - \measuredangle XYY'' = \measuredangle YXM - \measuredangle YXX'' = \measuredangle X''XM$. Furthermore, obviously the angles $\measuredangle YY''M$ and $\measuredangle XX''M$ are equal (both are 90°), and $YM = XM$. This shows that $\Delta YY''M \cong \Delta XX''M$, and $MY'' = MX''$. In other words, the distance from M to the perpendicular to BC through Y equals the distance from M to the perpendicular to AB through X. Similarly, this distance equals the distance from M to the perpendicular to CD through Z, and the latter equals to the distance from M to the perpendicular to DA through W. Hence, the point M indeed is equidistant from the four perpendiculars to AB, BC, CD, DA through X, Y, Z, W. This proves Theorem 2.

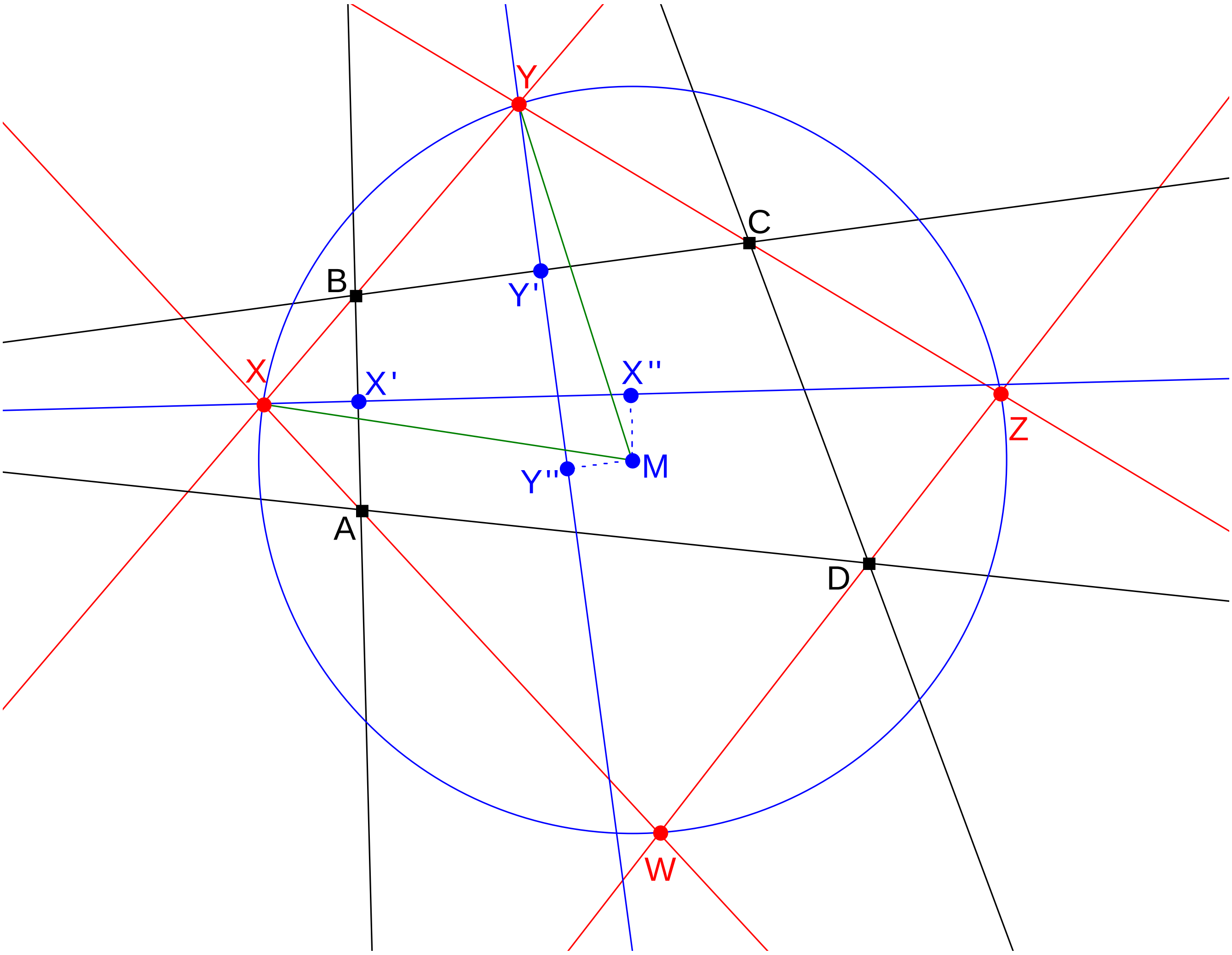

Fig. 4

§3. The circumscribed quadrilateral

Now we restrict ourselves to a special kind of quadrilaterals. Namely, let *ABCD* be a circumscribed quadrilateral and *O* its incenter. [A *circumscribed quadrilateral* is a quadrilateral having an incircle.] Then we have (Fig. 5):

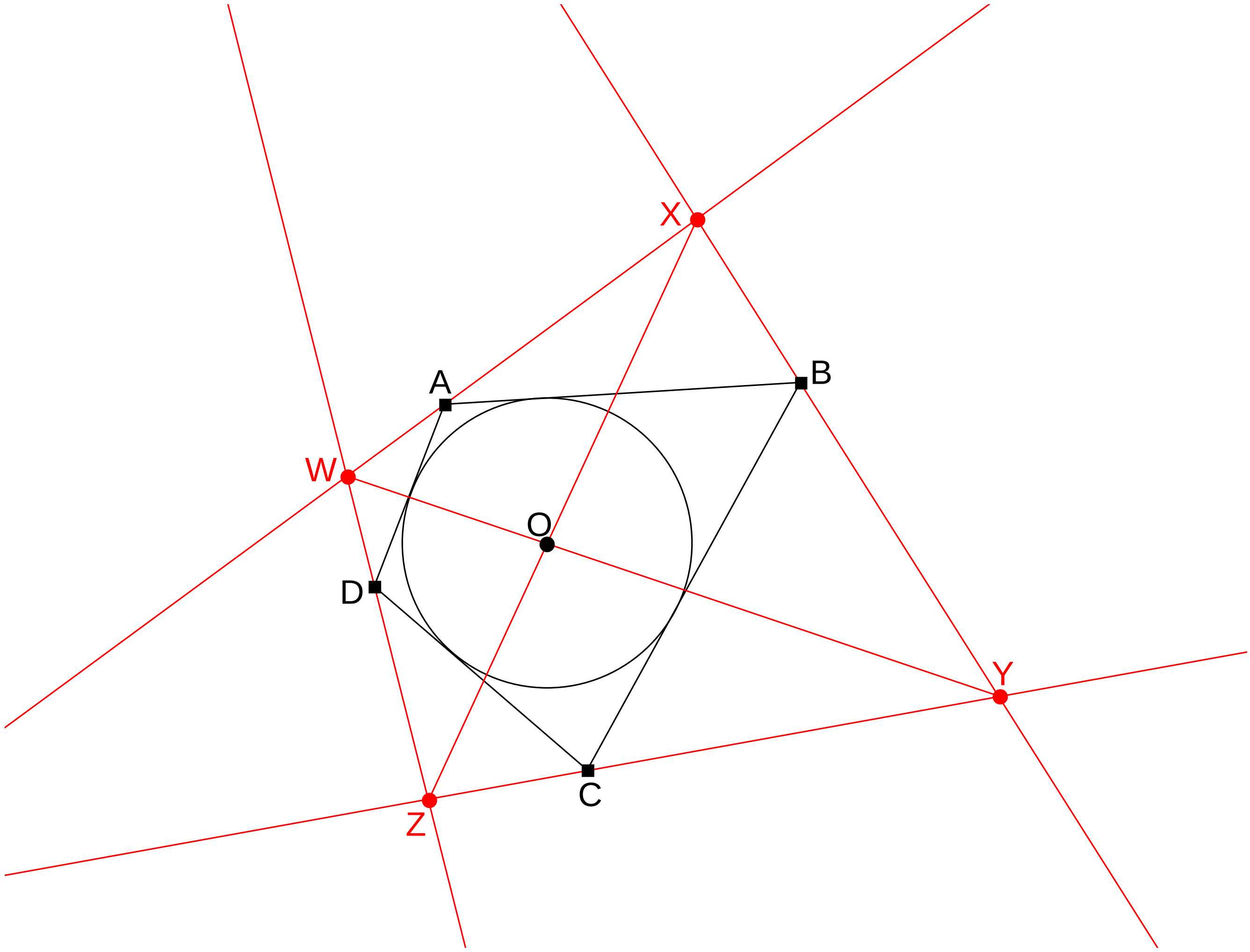

Fig. 5

Theorem 3. The diagonals *XZ* and *YW* of the quadrilateral *XYZW* meet at *O*.

Proof (Fig. 6). If *AB* and *CD* meet at *E*, then the rays *EA* and *ED* form the angle *AED*. These rays are touched by the excircles of the quadrilateral *ABCD* centered at *W* and *Y* on one hand, and by the incircle of quadrilateral *ABCD* centered at *O* on the other hand. Hence, the centers *W*, *Y*, *O* lie on the angle bisector of the angle *AED*, i. e., the point *O* lies on *YW*. Similarly, *O* lies on *XZ*, and Theorem 3 is proven.

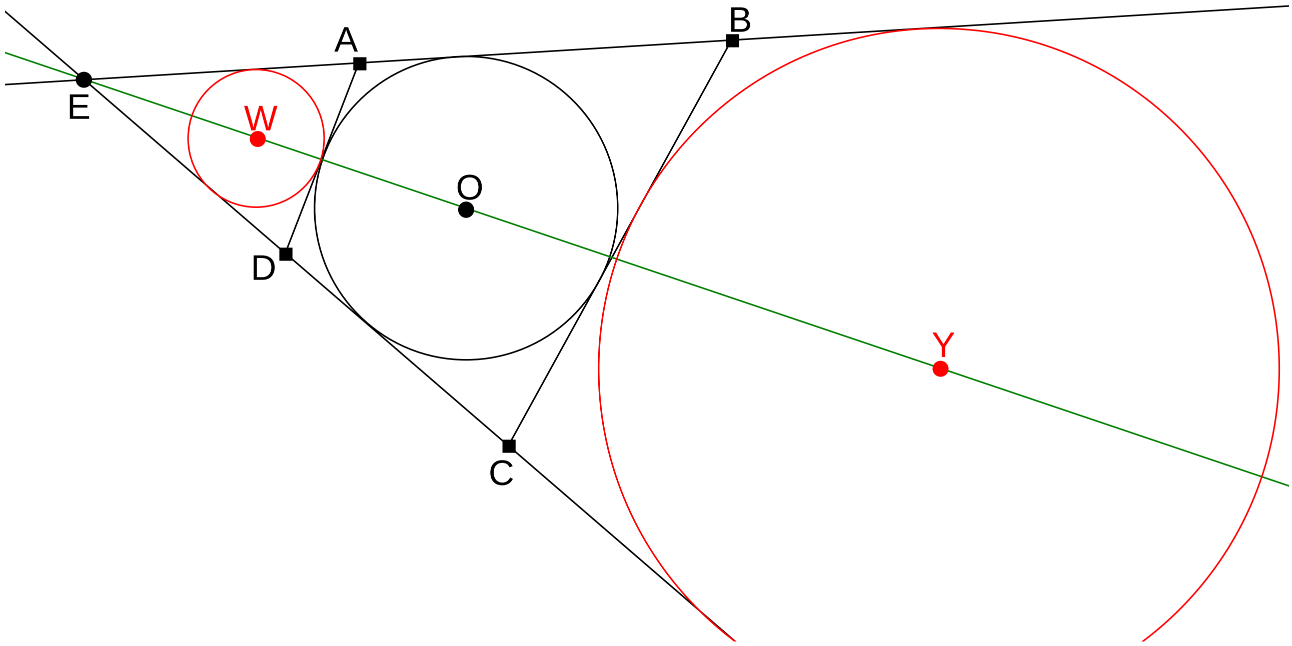

Fig. 6

In the following, we will use the shorthand notation *perpenbisector* for *perpendicular bisector*.

Let x, y, z, w be the perpendiculars to AB, BC, CD, DA through X, Y, Z, W. Theorem 2 states that these lines x, y, z, w are equidistant from M, i. e. that $d(M; x) = d(M; y) = d(M; z) = d(M; w)$, where $d(P; g)$ signifies the distance from a point P to a line g. Let $d = d(M; x) = d(M; y) = d(M; z) = d(M; w)$.

The midpoint of the segment OM will be denoted by M' (Fig. 7).

Since the incircle of $ABCD$ touches the four segments AB, BC, CD, DA, its center O lies on the internal angle bisectors of the angles DAB, ABC, BCD, CDA.

Since the internal and the external angle bisectors of an angle are perpendicular to each other, we have $\measuredangle OCZ = 90°$ and $\measuredangle ODZ = 90°$, so that C and D lie on the circle with diameter OZ. Consequently, the perpenbisector of the segment CD passes through the center of this circle, i. e. through the midpoint Q of OZ.

Let x', y', z', w' be the perpenbisectors of the segments AB, BC, CD, DA. Since $z \perp CD$ and $z' \perp CD$, we have $z' \parallel z$. Hence, the perpenbisector z' of CD is parallel to z and passes through Q.

The homothety with center O and factor $\frac{1}{2}$ maps Z to the midpoint Q of OZ; hence, the line z passing through Z is mapped to the parallel to z through Q, i. e. to the line z'. On the other hand, obviously, the image of M in this homothety is the midpoint M' of OM. Hence, after the fundamental properties of homotheties, $d(M'; z') = \frac{1}{2} \bullet d(M; z)$, i. e. $d(M'; z') = \frac{1}{2}d$. Similarly, $d(M'; x') = d(M'; y') = d(M'; w') = \frac{1}{2}d$. Hence, the point M' is equidistant from the perpenbisectors x', y', z', w' of AB, BC, CD, DA. We can state the following:

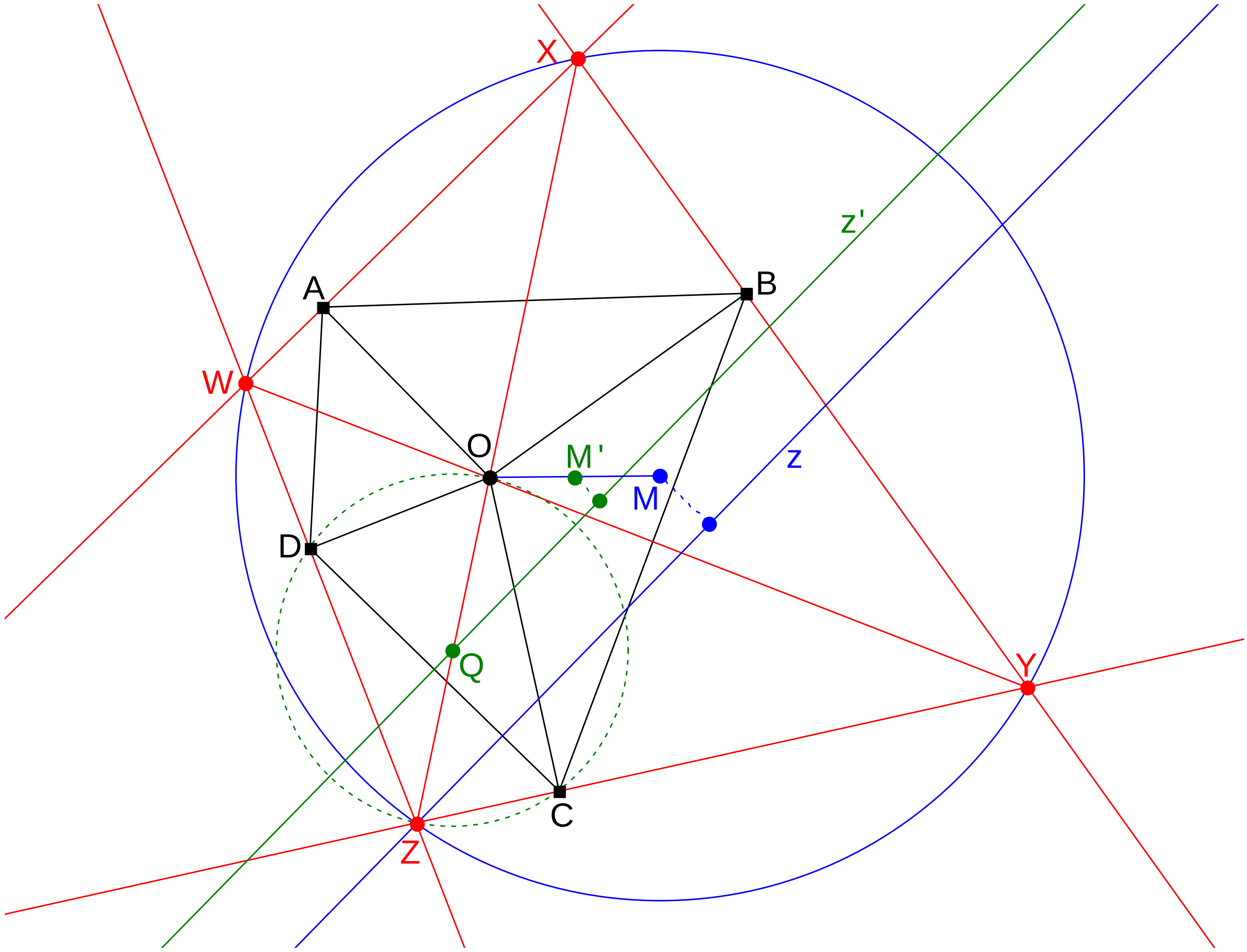

Fig. 7

Theorem 4. The midpoint M' of the segment OM is equidistant from the perpenbisectors of the segments AB, BC, CD, DA. In other words, there exists a circle centered at M' and touching the perpenbisectors of the segments AB, BC, CD, DA (Fig. 8).

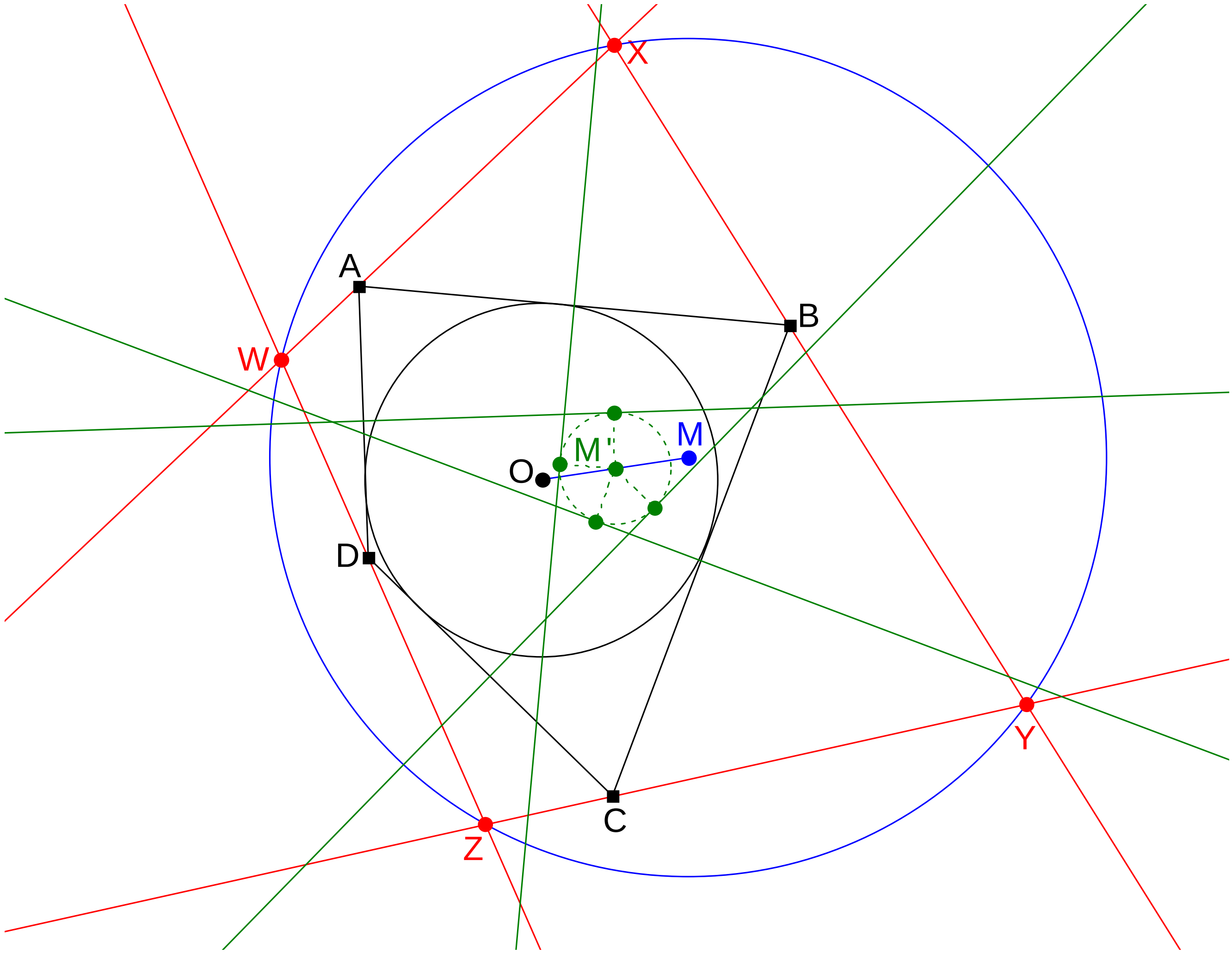

Fig. 8

§4. PB-quadrilaterals

Let *ABCD* again be any quadrilateral. The perpenbisectors of the segments *DA* and *AB* meet at A'. The perpenbisectors of the segments *AB* and *BC* meet at B'. The perpenbisectors of the segments *BC* and *CD* meet at C'. The perpenbisectors of the segments *CD* and *DA* meet at D'.

The quadrilateral $A'B'C'D'$ will be called the *PB-quadrilateral* of the quadrilateral *ABCD*. ("PB" is a shorthand for "perpenbisectors".) (See Fig. 9.)

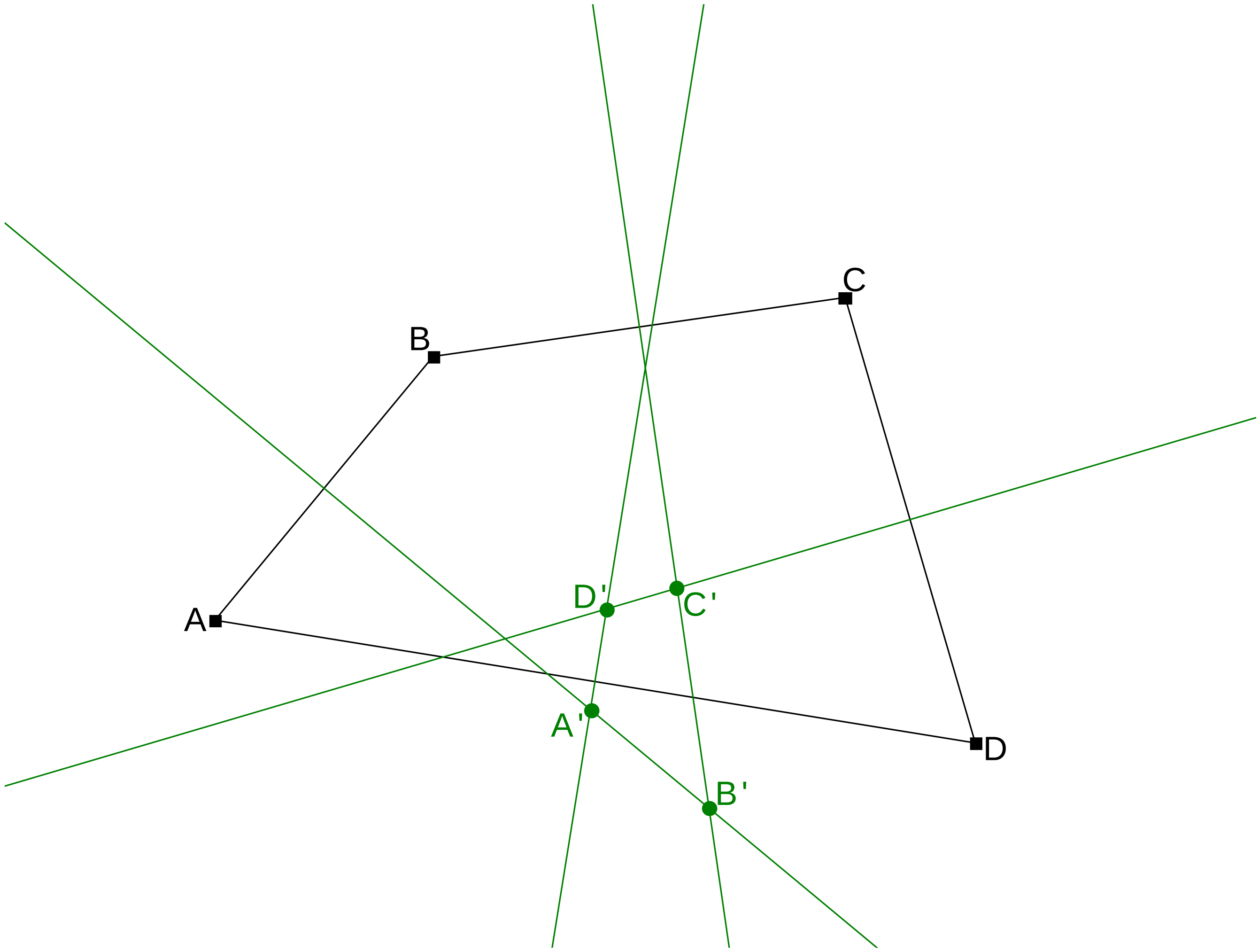

Fig. 9

Obviously, the quadrilateral $A'B'C'D'$ degenerates to a point if the quadrilateral *ABCD* is inscribed (in fact, the perpenbisectors of the sides of an inscribed quadrilateral all pass through the circumcenter). On the other hand, we are going to consider the case when the quadrilateral *ABCD* is circumscribed. Then, Theorem 4 secures the existence of a circle touching the perpenbisectors of *AB*, *BC*, *CD*, *DA*. In other words, the quadrilateral $A'B'C'D'$ formed by these perpenbisectors is circumscribed. Hence, we may state:

Theorem 5. The PB-quadrilateral of a circumscribed quadrilateral is a circumscribed quadrilateral again (Fig. 10).

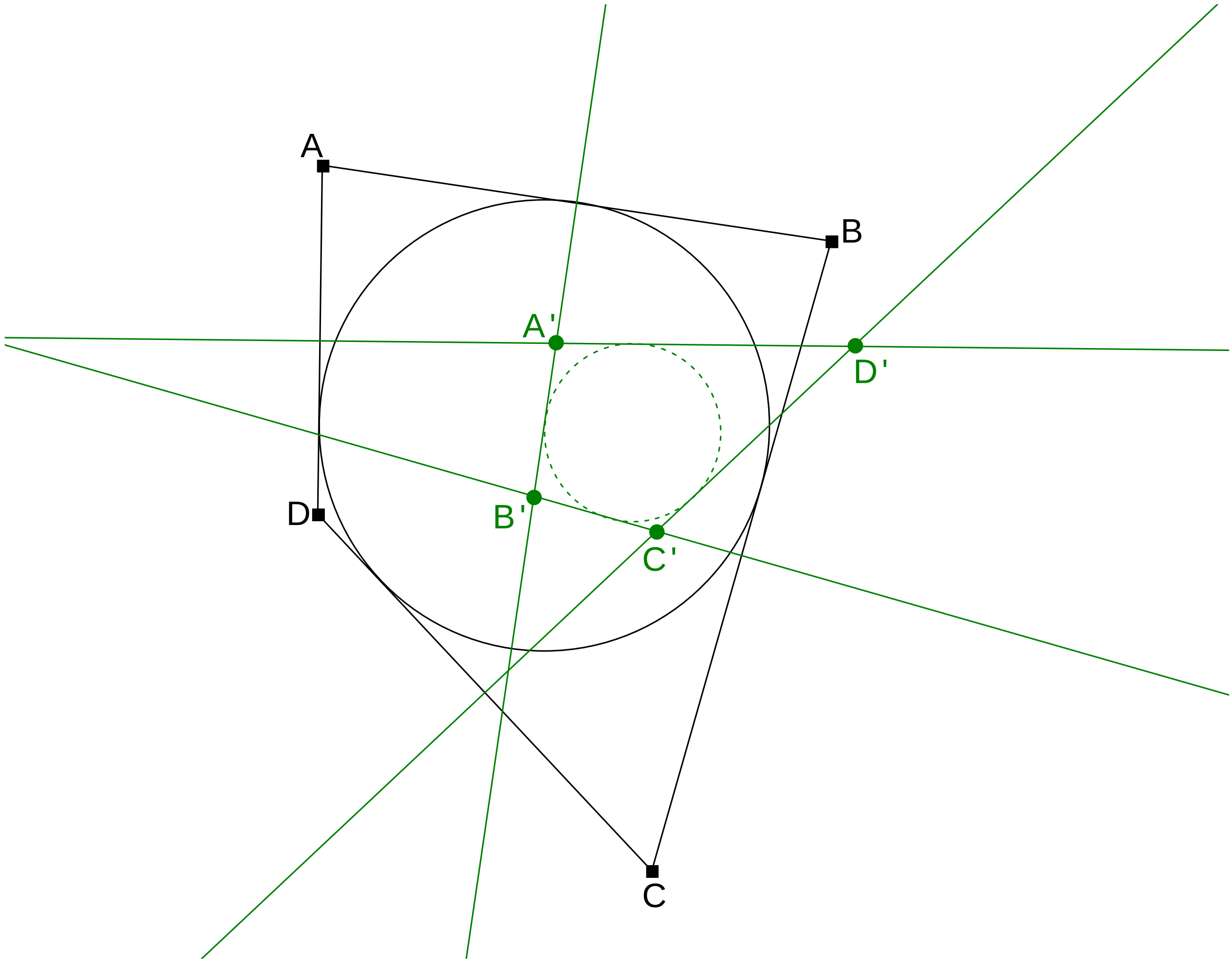

Fig. 10

This result has a kind of converse:

Theorem 6. If *ABCD* is an arbitrary, but not inscribed, quadrilateral, and if the PB-quadrilateral of *ABCD* is circumscribed, then the quadrilateral *ABCD* is circumscribed, too.

§5. Proof of the Converse

In order to prove Theorem 6, we will make use of a lemma related to the ”second generation” of the PB-quadrilateral, i. e. the PB-quadrilateral of the PB-quadrilateral:

Theorem 7. If $A'B'C'D'$ is the PB-quadrilateral of an arbitrary quadrilateral *ABCD*, and $A''B''C''D''$ is the PB-quadrilateral of $A'B'C'D'$, then the quadrilaterals *ABCD* and $A''B''C''D''$ are homothetic (Fig. 11).

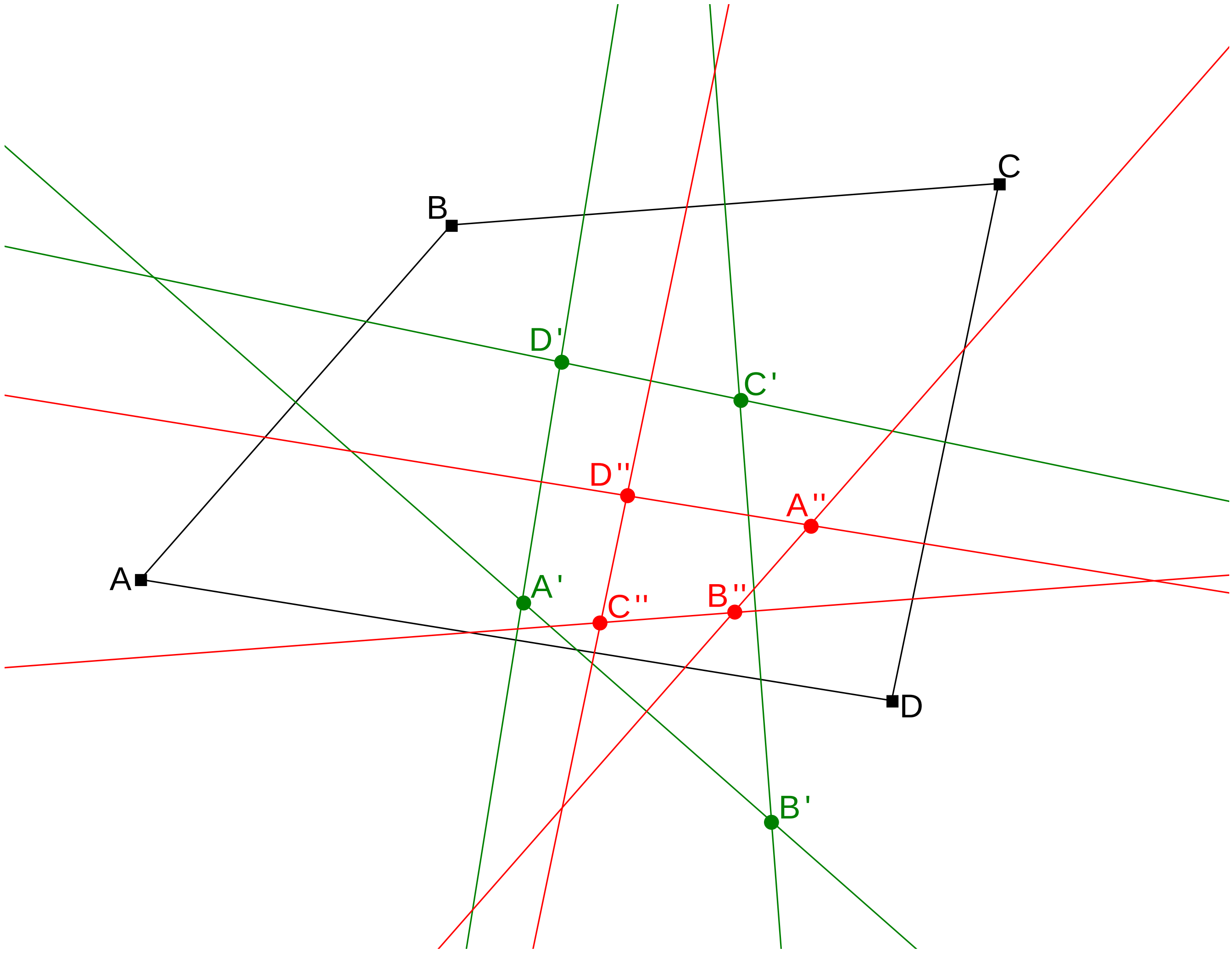

Fig. 11

Proof. We begin by considering $A'B'C'D'$. The line $A'B'$ is the perpenbisector of AB; hence, $A'B' \perp AB$, and similarly, $B'C' \perp BC$, $C'D' \perp CD$ and $D'A' \perp DA$.

The point A' is the meet of the perpenbisectors of DA and AB, i. e. of the perpenbisectors of two sides of ΔDAB. Hence, A' also lies on the perpenbisector of the third side BD. Similarly, C' lies on this perpenbisector. Hence, $A'C'$ is the perpenbisector of BD, and $A'C' \perp BD$. Similarly, $B'D' \perp AC$.

The same can be applied to $A''B''C''D''$ to obtain $A''B'' \perp A'B'$, $B''C'' \perp B'C'$, $C''D'' \perp C'D'$, $D''A'' \perp D'A'$, $A''C'' \perp B'D'$ and $B''D'' \perp A'C'$.

From $A''B'' \perp A'B'$ and $A'B' \perp AB$, we infer $A''B'' \parallel AB$; similarly, $B''C'' \parallel BC$, $C''D'' \parallel CD$ and $D''A'' \parallel DA$. From $A''C'' \perp B'D'$ and $B'D' \perp AC$, we conclude $A''C'' \parallel AC$; likewise, $B''D'' \parallel BD$.

Therewith, we have obtained six parallelisms in total: Any side or diagonal of the quadrilateral $A''B''C''D''$ is parallel to the corresponding side or diagonal of the quadrilateral $ABCD$. Now, from this, the homothety of the quadrilaterals $ABCD$ and $A''B''C''D''$ can be inferred as follows: Since $A''B'' \parallel AB$, $B''C'' \parallel BC$, $A''C'' \parallel AC$, the triangles ABC and $A''B''C''$ are homothetic; let Z be their homothetic center. Then, a homothety with center Z and factor $ZC'' : ZC$ (with directed lengths) transforms ΔABC to $\Delta A''B''C''$. The homothetic center Z is the intersection of the lines AA'', BB'', CC''.

Similarly, there is a homothety with a center Z' and factor $Z'C'' : Z'C$ transforming ΔCDA to $\Delta C''D''A''$; hereby, the homothetic center Z' is the intersection of the lines CC'', DD'', AA''.

Now, the points Z and Z' coincide, both of them being intersections of AA'' and CC''; hence, the factors $ZC'' : ZC$ and $Z'C'' : Z'C$ are equal, too. Therefore, there is one and the same homothety transforming ΔABC to $\Delta A''B''C''$ and ΔCDA to $\Delta C''D''A''$, and consequently, this homothety will map the quadrilateral *ABCD* to the quadrilateral $A''B''C''D''$. Hence, the two quadrilaterals are homothetic, proving Theorem 7.

This proof of Theorem 7 is indicated quite laconically in [1], Satz 1.3.2.

Let's finally conclude with the *proof of Theorem* 6. Assume the quadrilateral *ABCD* is not inscribed, and its PB-quadrilateral $A'B'C'D'$ is circumscribed. To show that *ABCD* is circumscribed, too.

According to Theorem 7, the PB-quadrilateral $A''B''C''D''$ of $A'B'C'D'$ is homothetic to *ABCD*. On the other hand, since the quadrilateral $A'B'C'D'$ is circumscribed, Theorem 5 yields that its PB-quadrilateral $A''B''C''D''$ is also circumscribed, and hence, the quadrilateral *ABCD* homothetic to $A''B''C''D''$ is circumscribed, too. This proves Theorem 6.

References

[1] Wilfried Haag: *Wege zu geometrischen Sätzen*, 1st Edition Stuttgart 2003.

[2] Jacques Hadamard: *Leçons de géométrie élémentaire I*: *Géométrie plane*, Paris 1911.

[3] Marcello Tarquini: *Hyacinthos message* #9011, 2004.

[4] Darij Grinberg: *Die vier Ankreise eines Dreiecks*, $\sqrt{\text{WURZEL}}$ 2/2002, 26-32.