Workshop presented at AMESA Congress, 30 June -4 July 2004, Univ. of North-West, Potchefstroom and at NCTM Annual Meeting, 21-24 April 2004, Philadelphia, PA. Published in *AMESA Congress Proceedings*, 2004, Vol. 2, pp. 52-56.

# CLOUGH'S CONJECTURE: A SKETCHPAD INVESTIGATION

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**Target Audience**: GET and FET **Duration**: 2 hours

**Required**: Computer Lab with *Sketchpad* 4 plus data projector & OHP

**Max. Participants**: Dependent on no. of computers - max. 2 per computer

**Accompanying Sketchpad Sketch**:

http://mysite.mweb.co.za/residents/profmd/clough.zip

This workshop will start with a brief introduction to Viviani's theorem (see De Villiers, 2003), followed by a *Sketchpad* investigation of a directly related, and possibly new conjecture by a Grade 11 learner from Bishops in Cape Town. The activity provides a useful opportunity for engaging participants in conjecturing, logical explanation, generalising, refutation, and verification. During the activity some of the roles played by proof in mathematics such as verification, explanation, discovery and intellectual challenge will also be briefly discussed.

The activity involves the investigation of a surprising property related to an equilateral triangle, and generalisations of the result to other types of polygons. Extensive use will be made of the ability of *Sketchpad* to provide dynamic visual stimuli to check and refute conjectures, and look for possible clues for logical explanations.

Overall the activity is intended to elicit some mathematical surprises and feelings such as "*How come? Why? I don't believe it - WOW!*" Participants will be guided through both geometric and algebraic explanations. Time permitting, participants might also be guided to another interesting variation related to the same problem.

## Reference

De Villiers, M. (2003). *Rethinking Proof with Sketchpad 4*, Key Curriculum Press, Emeryville: USA.

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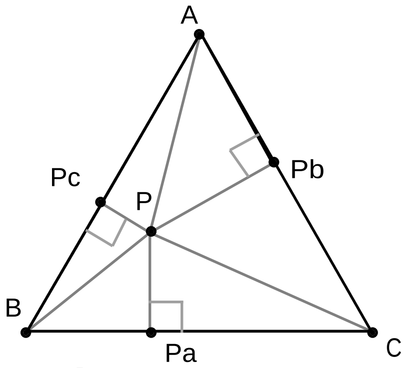
# Clough's Conjecture




## Name(s)______

During 2003, Duncan Clough, a Grade 11 student from Bishops in Cape Town was exploring Viviani's theorem, which says that the sum of distances of a point to the sides of an equilateral triangle is constant (see the Distances activity from *Rethinking Proof*). Labeling the feet of the altitudes from an arbitrary point $P$ inside an equilateral triangle $ABC$ to the sides $AB$, $BC$, $AC$ respectively as $P_c$, $P_a$, $P_b$, he next measured the distances $AP_c$, $BP_a$ and $CP_b$. What interesting observation did he make?

### CONJECTURE

- Open the sketch **clough.gsp**[1] (and click on *conjecture* tab if necessary). Drag point $P$ to experiment with your sketch.

**1**. Press the button to show the distance sum. Drag point $P$ around the interior of the triangle. What do you notice about the sum of the distances?

**2**. Drag a vertex of the triangle to change the triangle's size. Then again drag point $P$ around the interior of the triangle. What do you notice now?

**3**. What happens if you drag $P$ far outside the triangle?

**4**. Drag $P$ to a vertex point. How is the sum of these distances related to the sides of the original triangle?

- Check your observation in Question 4 by pressing the sum-side relationship button.

**5**. Organize your observations from Questions1-4 into a conjecture. Write your conjecture using complete sentences.

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[1] Download zipped *Sketchpad* sketch from http://mysite.mweb.co.za/residents/profmd/clough.zip

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# EXPLAINING

## Challenge

Use another sheet of paper to try to logically explain your conjecture from Question 5 above. If you get stuck, use the steps and hints that follow to develop an explanation of your conjectures.

*Geometric Explanation*

- Click on the *geometric explanation* tab. Press the button to respectively draw perpendiculars to *AB* at *A*, *BC* at *B* and *CA* at *C*.

**6**. What type of triangle *KLM* is formed by the three perpendiculars?

**7**. What are the sizes of angles *KAB* and *KBA*? Why?

**8**. What is the size of angle *BKA*? Why?

**9**. Calculate the other angles of triangle *KLM* to explain your observation in Question 6.

**10**. What connection is there between $AP_C$ and the distance from *P* to side *KM*? Why?

**11**. Use your observation in Question 10 and Viviani's theorem to complete your explanation of Clough's Conjecture so that it can achieve the status of a Theorem.

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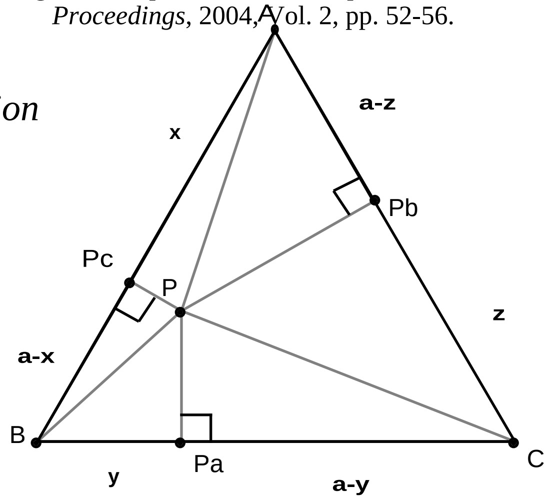
*Algebraic Explanation*




Let $AB = a$, $AP_c = x$, etc. as shown.

**12**. Apply the theorem of Pythagoras to hypotenuse *AP* in triangle $APP_c$.

**13**. Similarly, apply the theorem of Pythagoras to hypotenuse *AP* in triangle $APP_b$, and equate this to the expression in Question 13.

**14**. Write down two similar equations by applying the theorem of Pythagoras (in the same order and form) to the right triangles adjacent to hypotenuses BP and CP.

**15**. Add up the left and right hand sides of these three equations and simplify to conclude the explanation.

**Present Your Explanations**

Fully write out your explanations as arguments in paragraph form, or as two column proofs.

**Further Exploration**

**1**. a) Click on the *rhombus* tab and drag point *P* around. What do you observe?

b) Explain your observation in 1a), and generalize to polygons with a similar property.

**2**. a) Click on the *parallelogram* tab and drag point *P* around. What do you observe?

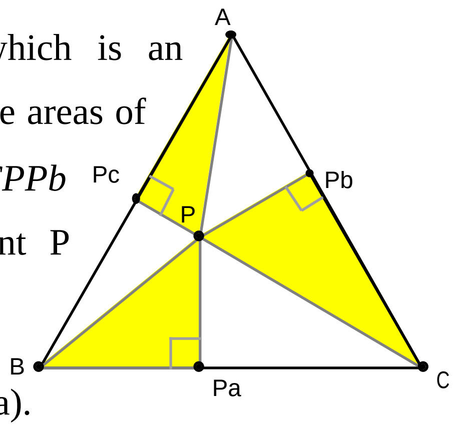
b) Explain your observation in 2a), and generalize to polygons with a similar property.

**3.** a) Click on the *pentagon* tab which is a dynamic sketch of a pentagon with all angles equal, and drag point *P* around. What do you observe?

b) Explain your observation in 3a), and generalize to polygons with a similar property.

## A Surprise Variation

1. a) Click on the sum areas tab which is an equilateral triangle with the sum of the areas of triangles *APPc*, *BPPa* and *CPPb* measured and calculated. Drag point P around. What do you observe?




b) Explain your observation in 1a). Click on the provided hint if necessary.

2. Explore the general type of polygon for which this area result holds and develop an explanation.