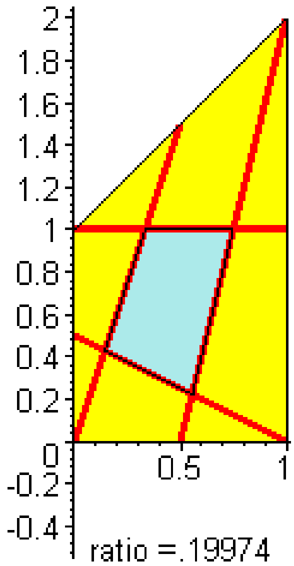
An interesting quadrangle problem

This problem appears in *Rethinking Proof with the Geometers Sketchpad*, by Michael de Villiers. It provides a good example of how programs like Maple and Geometers Sketchpad can be used to experiment and conjecture, then even assist with the calculations needed to establish the conjecture (if it is true).

Problem: Given a convex quadrangle $Q = A_1 A_2 A_3 A_4$, with the vertices labelled clockwise, let $IQ = B_1 B_2 B_3 B_4$ denote the **clockwise interior quadrangle** of Q, that is the (necessarily convex) quadrangle whose vertices are the intersections of the four 'clockwise' medians of Q, $A_i M_i$, where M_i is the midpoint of the segment $A_{i+1} A_{i+2}$ for i = 1, 2 , M_3 is the midpoint of $A_4 A_1$, and M_4 is the midpoint of $A_1 A_2$.

Determine bounds on the ratio $\dfrac{\mathrm{area}(B_1 B_2 B_3 B_4)}{\mathrm{area}(A_1 A_2 A_3 A_4)}$

Here is a picture of a typical Q with its clockwise interior quadrangle drawn in blue. The ratio is calculated and displayed below the picture.



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Solution. $\dfrac{1}{6} < \dfrac{\mathrm{area}(IQ)}{\mathrm{area}(Q)} \le \dfrac{1}{5}$ and the bounds are sharp.

Without loss of generality, we can assume that $A_1 = [0, 1]$, $A_2 = [0, 0]$, $A_3 = [1, 0]$ and $A_4 = [a, b]$ where a and b are positive and a+b > 1. This is because the quadrangle can be so situated with an affine map, which preserve ratios of areas.

Using determinants (see the Maple computations below), the ratio of the areas works out to $\dfrac{top}{bottom}$, where

top =

$$4a^4 + (12 + 22b)a^3 + (41b^2 - 19 + 27b)a^2 + (-11b + 8 + 28b^3 + 9b^2)a - 1 + 4b^4 - 4b + 6$$

and bottom = $(3a+1+b)(-1+4b+2a)(-2+3b+4a)(a+2+2b)$.

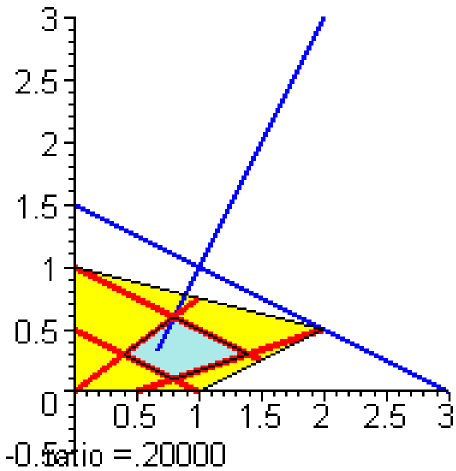
To establish the right hand inequality $\frac{\text{area}(IQ)}{\text{area}(Q)} \le \frac{1}{5}$ **,**we need to show $bottom - 5\,top$ is never negative. But when factored (see the Maple computations below), this quantity is

$$(a+2b-3)^2(2a-b-1)^2$$

This is always >= 0 with equality when $b = \frac{3}{2} - \frac{a}{2}$ for 0 < a < 3 or $b = 2a - 1$ for $\frac{2}{3} < a$.

Hence the right hand inequality is sharp, in the sense that equality is achieved and no smaller upper bound is possible.

Here is a picture showing the segment $b = \frac{3}{2} - \frac{a}{2}$ for 0 < a < 3 and the ray $b = 2a - 1$ for $\frac{2}{3} < a$



In the case $b = 2a - 1$ (i. e., (a,b) is on the blue ray with positive slope), we can derive that $\frac{b+1}{a} = \frac{b}{a - \frac{1}{2}}$ which says that the sides of the interior quadrangle with positive slope are parallel. In the case $b = \frac{3}{2} - \frac{a}{2}$, (i.e., (a,b) is on the blue segment with negative slope), we can derive that $\frac{\frac{b}{2} - 1}{\frac{a+1}{2}} = -\frac{1}{2}$, which says that the sides of the interior quadrangle with negative slope are parallel. Hence the ratio is 1/5 when the interior quadrangle is a trapezoid.

To establish the left hand inequality $\frac{1}{6} < \frac{\text{area}(IQ)}{\text{area}(Q)}$, we need to show that $6\,top - bottom$ is always positive.

But when factored (see the Maple computations below), this quantity is

$$5(4a+ab-2+2b^2)(2a^2-3a+1+b+4ab)$$

The factor $4a+ab-2+2b^2$ is always positive for a > 0, b > 0, and $1 < a+b$: To see this,

$$4a+ab-2+2b^2 = 4a+b(a+b)-2+b^2$$

$$> 4a+b-2+b^2$$

$$> 3a-1+b^2$$

$$> 4-3b+b^2$$

$$> 4-4b+b^2 = (2-b)^2 > 0 \text{ except when } b = 2$$

But in that case (b = 2), $4a+ab-2+2b^2 = 4a+2a-2+8 = 6a+6 > 0$

Note that when b = 1, a = 0, the quadrangle collapses to a triangle and the factor is 0.

The factor $2a^2-3a+1+b+4ab$ is also always positive for a > 0, b > 0, 1 < a+b:

To see this,

$$2a^2-3a+1+b+4ab = a^2-2a+1+b+3ab+a(a+b)$$

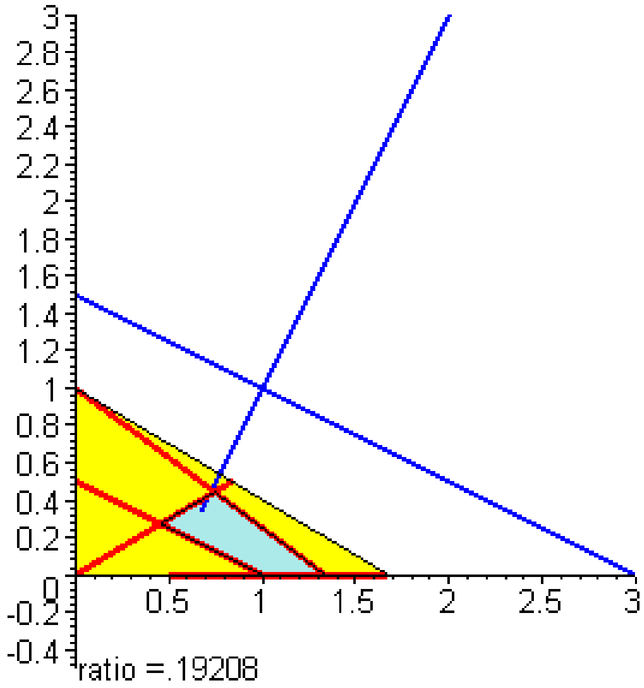
$$> a^2-2a+1+b+3ab = (a-1)^2+b+3ab > 0.$$

When a = 1 and b = 0, we get the other degenerate case where the ratio becomes 1/6. The left hand inequality is sharp also, in the sense that there are quadrangles whose area ratio is as close to 1/6 as you want, but the ratio 1/6 is never achieved for a quadrangle.

Since affine maps preserve parallel lines, the 'parallel property' established above is preserved when the normalized quadrangle is 'un-normalized' back to the original, so we have a theorem.

Theorem: The clockwise interior quadrangle is 1/5 of its exterior quadrangle precisely when it is a trapezoid, otherwise it is less than a 1/5 but greater than a 1/6.

Here is an animation which illustratest the theorem.



Note: The same theorem holds if clockwise is replaced by 'counterclockwise' in the above theorem.

Analogous Problems:

1. Ask the analogous question about convex pentagons (hexagons, etc)

2. Let DEF be the triangle interior to ABC formed by joining A to 1/3 B + 2/3 C, B to 1/3 C + 2/3 A, and C to 1/3 A + 2/3 B. Find bounds on area(DEF)/area(ABC).

Maple computations