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Conjecture and Proof with *Sketchpad*: A case study

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Stimulus

While walking through an academic bookstore in Berkeley, San Francisco earlier this year, I came upon the following interesting problem in a book:

> Prove that if P is any arbitrary point on the incircle of an equilateral ΔABC, then $PA^2 + PB^2 + PC^2$ is a constant.

In other words, the sum of the squares of the distances from P to the vertices is constant (see Figure 1).

Conjecture

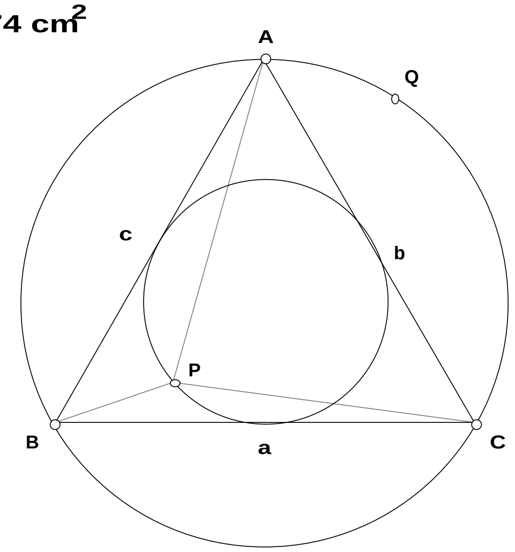
Due to an often observed (although not generally valid) duality between *incircles* and *circumcircles*, as well as between *vertices* and *sides* in plane geometry (for example see De Villiers, 1992, 1993a, 1993b, 1996), I intuitively anticipated the following dual result:

> If Q is any arbitrary point on the circumcircle of an equilateral triangle, then $h_{AB}^2 + h_{BC}^2 + h_{AC}^2$ is a constant (where h_i are the distances from Q to the three sides).

Investigation of this conjecture with the dynamic geometry programme *Sketchpad*[1] where Q can be **moved** along the circle while measurements are continuously updated, quickly confirmed it (see Figure 1 - for this sketch the sum remains equal to 38.84 no matter how Q is moved). (An extremely useful feature of *Sketchpad* (and other similar dynamic geometry software) is that calculations can be done, and displayed on all measurements). Furthermore, as shown by the last calculation in Figure 1, the ratio between the constant for the incircle to that of the circumcircle is 1.67. My guess at this stage was that this ratio was probably exactly equal to $1\frac{2}{3}$ as a setting of the measurement and calculation accuracy of *Sketchpad* to thousandths gave 1.667.

Distance Q to a = 5.58 cm
Distance Q to b = 1.61 cm
Distance Q to c = 2.25 cm
$(\text{Distance Q to a})^2 + (\text{Distance Q to b})^2 + (\text{Distance Q to c})^2 = 38.84 \text{ cm}^2$

PA = 5.75 cm
PB = 2.16 cm
PC = 5.20 cm
$PA^2 + PB^2 + PC^2 = 64.74 \text{ cm}^2$



$$\frac{PA^2 + PB^2 + PC^2}{(\text{Distance Q to a})^2 + (\text{Distance Q to b})^2 + (\text{Distance Q to c})^2} = 1.67$$

Figure 1

Investigate these theorems dynamically with the use of *Sketchpad!*

1. If you do not have Sketchpad, first download a **FREE DEMO** of it from:
   http://www.keypress.com/sketchpad/sketchdemo.html
2. Then download the following to dynamically investigate these theorems:
   http://mzone.mweb.co.za/residents/profmd/conject.zip

Proof of first result

Consider Figure 2 where the equilateral ΔABC has been placed with its incentre/circumcentre at (0;0). Without loss of generality, let A($0;\ 2a$), B($-\sqrt{3}a;\ -a$), C($\sqrt{3}a;\ -a$) and P($x;\ y$). Then the incircle is $x^2 + y^2 = a^2$ and $PA^2 + PB^2 + PC^2$

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$$= \left[x^2 + (y-2a)^2\right] + \left[(x+\sqrt{3}a)^2 + (y+a)^2\right] + \left[(x-\sqrt{3}a)^2 + (y+a)^2\right]$$
$$= 3(x^2 + y^2) + 12a^2$$
$$= 3a^2 + 12a^2$$
$$= 15a^2$$

Since a is constant, it follows that the sum of the squares of the distances from P to the vertices must also be constant.

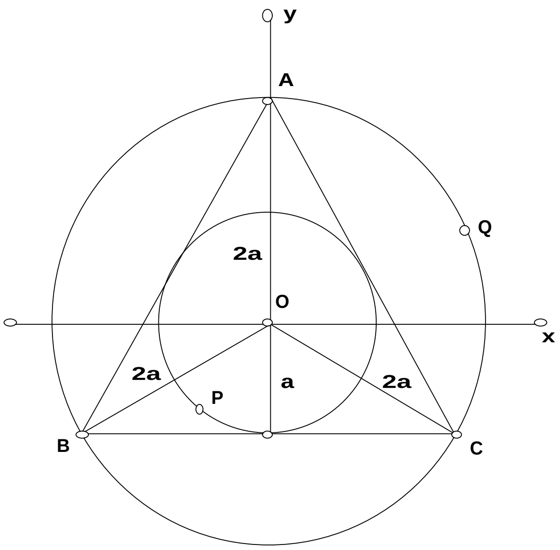

Figure 2

Proof of conjecture

To prove this conjecture, one needs to know (or deduce) what the squared distance is from a point (p;q) to a given line $y = mx + c$, which is given as:

$$\left(p - \frac{mq + p - cm}{m^2 + 1}\right)^2 + \left(q - \frac{c + m^2 p + mp}{m^2 + 1}\right)^2$$

If we consider the same triangle as in Figure 2, then the circumcircle is $x^2 + y^2 = 4a^2$, AB is $y = \sqrt{3}x + 2a$, AC is $y = -\sqrt{3}x + 2a$ and BC is $y = -a$. Then using the above result we have $h_{AB}^2 + h_{BC}^2 + h_{AC}^2$

$$
\begin{aligned}
&= \left(x - \frac{\sqrt{3}y + x - 2\sqrt{3}a}{4} \right)^2 + \left(y - \frac{2a + 3y + \sqrt{3}x}{4} \right)^2 \\
&\quad + (x - x)^2 + (y - (-a))^2 + \left(x - \frac{-\sqrt{3}y + x + 2\sqrt{3}a}{4} \right)^2 \\
&\quad + \left(y - \frac{2a + 3y - \sqrt{3}x}{4} \right)^2 \\
&= 3a^2 + \frac{3}{2}\left(x^2 + y^2\right) \\
&= 3a^2 + \frac{3}{2} \times 4a^2 \\
&= 9a^2
\end{aligned}
$$

Since *a* is constant, it follows that the sum of the squares of the distances from Q to the sides must be constant. Furthermore, the ratio of these two constants can now clearly be seen to be $15a^2/9a^2 = \frac{5}{3} = 1\frac{2}{3} = 1.67$.

Looking back

Although I knew from the very beginning that the first result was obviously true (otherwise why ask to **prove** it), I did not know beforehand whether the dual conjecture was true or not. Now given such a situation where no outside authority (a book, article or colleague) tells one *in advance* that a particular result is true or false, what is one to do?

It would seem that either one must first explore it experimentally by making some constructions (for which a programme like *Sketchpad* is ideally suited) or one must immediately try to construct a proof. However, if the proof is not a simple straight forward matter (as was the case here), it seems far more sensible to first check experimentally whether something is really true before one embarks on the possibly tough journey of trying to prove it. (If it is false, your experimental exploration will hopefully produce a counter-example, saving you valuable time in trying to unsuccessfully prove it).

Conviction is often a **prerequisite** for proof

Contrary to the traditonal teaching approach where proof is only presented as a prerequisite for conviction, we actually see from the above, that conviction is frequently a prerequisite for proof. In a research situation one would certainly not try to prove a result which seems rather doubtful, and one would only attempt a proof once one has satisfied oneself that it seems reasonable.

The educational implications of the above are many. Firstly, the traditional approach tends to give the false perception that new mathematics is only discovered deductively, and that experimentation is taboo in (real) mathematics. Secondly, in the traditional approach pupils have no ownership over results since the teacher, textbook and/or syllabus are seen as the ultimate authorities dictating, not only the choice of results, but their validity as well.

It would seem that the only way to change this is to actually involve pupils in the processes of mathematical discovery and proof in a research type setting. This would mean designing learning activities which allow for experimental exploration, conjecturing, refuting, formulating and reformulating, as well as explaining (proof). In this situation the teacher's role must then obviously change from that of an authority to that of a facilitator of learning, allowing pupils to take ownership and responsibility for their own mathematical development.

Of course, any experimental exploration (e.g. construction and/or calculation) by hand is extremely tedious. Furthermore, such pencil and paper work is quite often relatively inaccurate and one could fail to see an important underlying pattern or observe an important counter-example. It therefore necessitates the use of appropriate technology in the classroom such as scientific and graphic calculators or computer programmes like *Sketchpad*, *Cabri* or *Cinderella*.

Note

[1]In South Africa, *Sketchpad* is available from Dynamic Math Learning, 8 Cameron Rd, Pinetown, 3610. Tel: 031-7083709 or 031-7029941 (Pearl de Villiers). Cell: 082-2295103 (Pearl de Villiers) e-mail: dynamiclearn@mweb.co.za

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Mathematical Digest = http://www.mth.uct.ac.za/~digest/

International Journal of Mathematical Education in Science & Technology = http://mamch-mac.lboro.ac.uk/ijmest.html

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"*... computer graphics refreshes a distinction* between **fact** *and* **proof**, *one that many mathematicians prefer not to acknowledge but that Archimedes described wonderfully in these words: 'Certain things first became clear to me by a mechanical method, although they had to be demonstrated by geometry* **afterwards** *because their investigation by the said mechanical method did not furnish an actual demonstration. But it is of course easier, when the method had previously given us some knowledge of the questions, to supply the proof than it is to find it without previous knowledge. This is a reason why, in the case of the theorems that the volumes of a cone and a pyramid are one-third of the volumes of the cylinder and prism (respectively) having the same base and equal height, the proofs of which Eudoxus was the first to discover, no small share of the credit should be given to Democritus who was the first to state the fact, though without proof'*."

- Benoit Mandelbrot (1992: 88) Fractals, the Computer and Mathematics Education. **Proceedings of ICME-7**, Quebec, Canada.