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Conjecturing, refuting and proving within the context of dynamic geometry

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"*Mathematics is about problems, and problems must be made the focus of a students mathematical life. Painful and creatively frustrating as it may be, students and their teachers should at all times be engaged in the process - having ideas, not having ideas, discovering patterns, making conjectures, constructing examples and counterexamples, devising arguments, and critiquing each other's work.*" – Lockhart (2002, p. 16)

Lockhart (2002, p. 2), a research mathematician, describes the present system of mathematics education at school as a 'nightmare' claiming that it destroys children's 'natural curiosity and love of pattern-making'. He goes further in his critique of school mathematics by saying that 'no actual mathematics' (p. 14) is being done at school, and in the 'place of discovery and exploration' (p. 15) there is only mindless drill and exercise of given rules and algorithms.

The traditional teaching of proof does not escape Lockhart's scythe either when he writes as follows (p. 22): "*The art of proof has been replaced by a rigid step-by step pattern of uninspired formal deductions. The textbook presents a set of definitions, theorems, and proofs, the teacher copies them onto the blackboard, and the students copy them into their notebooks. They are then asked to mimic them in the exercises. Those that catch on to the pattern quickly are the 'good' students.*"

Lochart's criticism of school mathematics as being a caricature of 'real' mathematics reverberates over the years in the work of many mathematicians, mathematics educators and philosophers such as Klein (1924), Freudenthal (1973), Lakatos (1976), Davis, Hersh & Marchisotto (1995), Goldenberg, Cuoco & Mark, (1998), Borwein (2012), etc. to name but a few.

Though not intended to be exhaustive at all, Table 1 summarizes some of the major differences between 'real' mathematics and the traditional practice of teaching school (and undergraduate) mathematics.

	Classroom practice	Mathematical practice
Exploration, experimentation & conjecturing	Learners are not given the opportunity to explore and to experiment, and are given the conjecture as *fait accompli* in the form "Prove that …"	Mathematicians explore, experiment and make conjectures on patterns or any invariance they observe. The situation is typically open-ended and divergent.
Formulation	Learners are provided with a well-formulated conditional statement "Prove that *p* implies *q*."	Mathematicians formulate conjectures into conditional statements themselves, "If *p*, then *q*."
Truth value	1. The truth of the result is implied by the instruction "Prove that …"; so learners know beforehand that the result is true. 2. Learners accept the truth of the result on the authority of the teacher and the textbook.	1. Mathematicians do NOT necessarily know beforehand whether a conjecture is true or not. 2. Mathematicians usually determine the truth of a conjecture themselves via both experimentation and proof.
Proving	1. Learners almost never engage in the refutation of false conjectures because the curriculum/textbook only focuses on true statements. 2. Learners are usually guided by various sub-steps or sub-problems towards an eventual proof.	1. Mathematicians both logically prove and refute conjectures. 2. Mathematicians have to rely on their own ingenuity to make a logical connection between the premise and the conclusion.

Table 1: Comparison of classroom and 'real' mathematics

The purpose of this paper is to describe a mathematical exploration recently undertaken by the authors that aims to highlight some of the main features of conjecturing, refutation and proving. It is not implied that this particular investigation is suitable for use in a typical high school classroom, but it is hoped that it will inspire practicing teachers to critically reflect on the 'mathematical authenticity' of their own classroom practice. These examples might also be of value in mathematics teacher education to engage students in some conjecturing, refutation and proof.

INITIAL CONJECTURES

This investigation followed on the two iterative construction procedures described in De Villiers (2014) where the iterated triangles converged towards an equilateral triangle. The reader is now invited to recreate the constructions described below with suitable dynamic geometry software or to go online and dynamically experience the following two investigations by going to the ready-made *JavaSketchpad* sketches at:

http://dynamicmathematicslearning.com/collinear-incentres-conjecture.html

INVESTIGATION 1: TANGENT POINTS OF INCIRCLE

Start with any ΔABC and its incircle and incentre I. Label the points where the circle touches the sides BC, CA and AB respectively as A_1, B_1 and C_1 as shown in Figure 1. Repeat the process with the new $\Delta A_1B_1C_1$ and determine its incentre I_1. Then repeat the process twice more. Connect I to I_3 with a straight line. What do you visually notice about the four incentres? Check by dragging vertices A, B or C. Can you make a conjecture? Can you prove or disprove your conjecture?

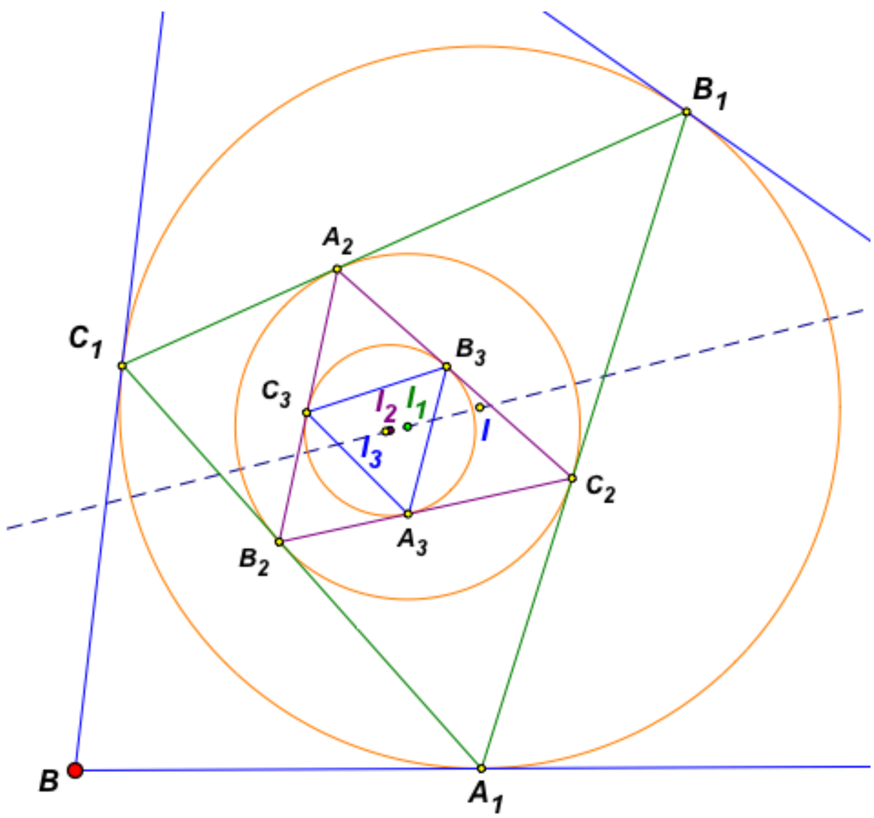

FIGURE 1: Iteration of tangent points of incircle.

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INVESTIGATION 2: EXCENTRES

Start with any ΔABC and construct its incentre I and excentres[1]. Label the excentres formed on the sides of the sides BC, CA and AB respectively as A_1, B_1 and C_1, and construct its incentre I_1. Repeat the process with the new $\Delta A_1B_1C_1$. Then repeat the process twice more. Connect I to I_3 with a straight line. What do you visually notice about the four incentres? Check by dragging vertices A, B or C. Can you make a conjecture? Can you prove or disprove your conjecture?

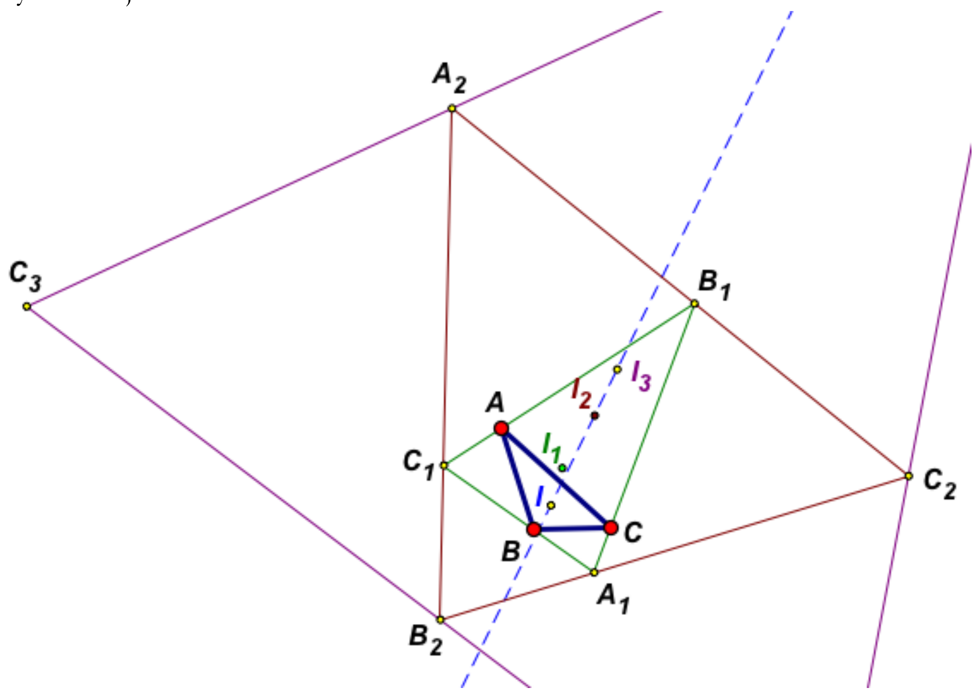

FIGURE 2: Iteration of excentres.

In Figure 1, it clearly seems that all four incentres are collinear (lie on the same straight line), with I_2 and I_3 almost coinciding. Likewise in Figure 2, though I_1 does not lie on the constructed line from I to I_3, the other three incentres do appear to be collinear. Checking of the conjectures by dragging within a dynamic geometry context convinced us that the conjectures were valid. The reader is now also encouraged to do so in the link provided earlier, if not already previously done.

Armed with compelling experimental evidence that our conjectures were true, having passed the so-called 'drag-test', we proceeded to attack the two conjectures trying both geometric as well as algebraic approaches. Neither approach was immediately successful with the algebraic approach especially becoming increasingly cumbersome and messy. Scanning the literature for any mention of the results, as well as other related mathematical results we might be able to use, also proved fruitless. However, we did find that Denison (2001) mentions the second

[1] The three excentres of a triangle are located at the intersection of the angle bisectors of the two exterior angles formed on each side of the triangle. For more information: http://en.wikipedia.org/wiki/Incircle_and_excircles_of_a_triangle

conjecture as unproved; however, incorrectly claiming that all these incentres are collinear, since I_1 does not lie on the line as already shown in Figure 2.

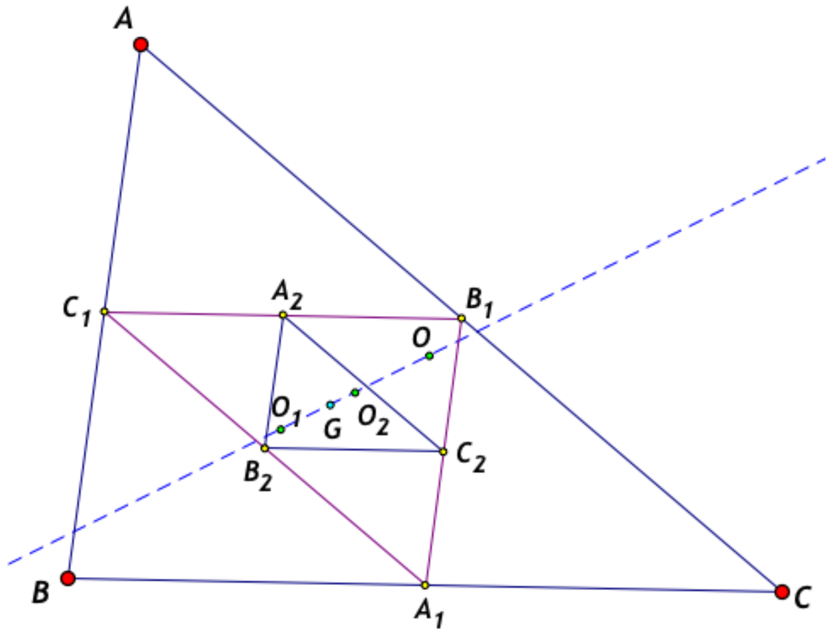

FIGURE 3: Iteration of circumcentres.

A THIRD CONJECTURE

In an effort to try and find possible ideas for proving our conjectures, we next considered similar constructions for the following cases:

1) Start with any ΔABC and construct its circumcentre O as shown in Figure 3. Label the midpoints of the sides BC, CA and AB respectively as A_1, B_1 and C_1, and construct its circumcentre O_1. Repeat the process with the new $\Delta A_1B_1C_1$. Then repeat the process once more. Connect O to O_1 with a straight line. What do you visually notice about the three circumcentres and the centroid G? Check by dragging vertices A, B or C. Can you make a conjecture? Can you prove or disprove your conjecture?[2]

2) Orthocentres of successive orthic triangles do not work.

3) Centroids of successive median triangles do not work either as one always gets the centroid of the original triangle.

PROOF OF CONJECTURE 3

Although the third conjecture about circumcentres above can be proved using coordinate geometry, it was more straightforward to prove it using the useful idea of a *spiral similarity*, which is defined as the composition of a rotation followed by a dilation (reduction or enlargement)[3]. For example, it is obvious that the median triangle $\Delta A_1B_1C_1$ is similar to ΔABC. Hence, a halfturn around the centre of similarity, the centroid G (the point of

[2] The reader is invited to also dynamically explore the above construction at: http://dynamicmathematicslearning.com/collinear-circumcentres-conjecture.html

[3] Read more about a spiral similarity at: http://www.cut-the-knot.org/Curriculum/Geometry/SpiralSimilarity.shtml

concurrency of the medians), followed by a dilation with scale factor ½ maps ΔABC onto $\Delta A_1B_1C_1$, and hence O onto O_1. Therefore, O, G and O_1 are collinear, and $GO = 2GO_1$. The same argument applies to the mapping of $\Delta A_1B_1C_1$ onto $\Delta A_2B_2C_2$; hence O_1, G and O_2 are collinear, and $GO_1 = 2GO_2$. By continuing the process, it follows that all further circumcentres will be collinear with G and the preceding circumcentres.

REFUTATION OF CONJECTURES 1 AND 2

Our frustrating inability to prove Conjectures 1 and 2 gradually led us to suspect that perhaps they were false, despite the seemingly convincing experimental evidence. So we went back to the proverbial drawing board to more closely examine the conjectures - this time looking more closely at them and trying to produce counter-examples to disprove them.

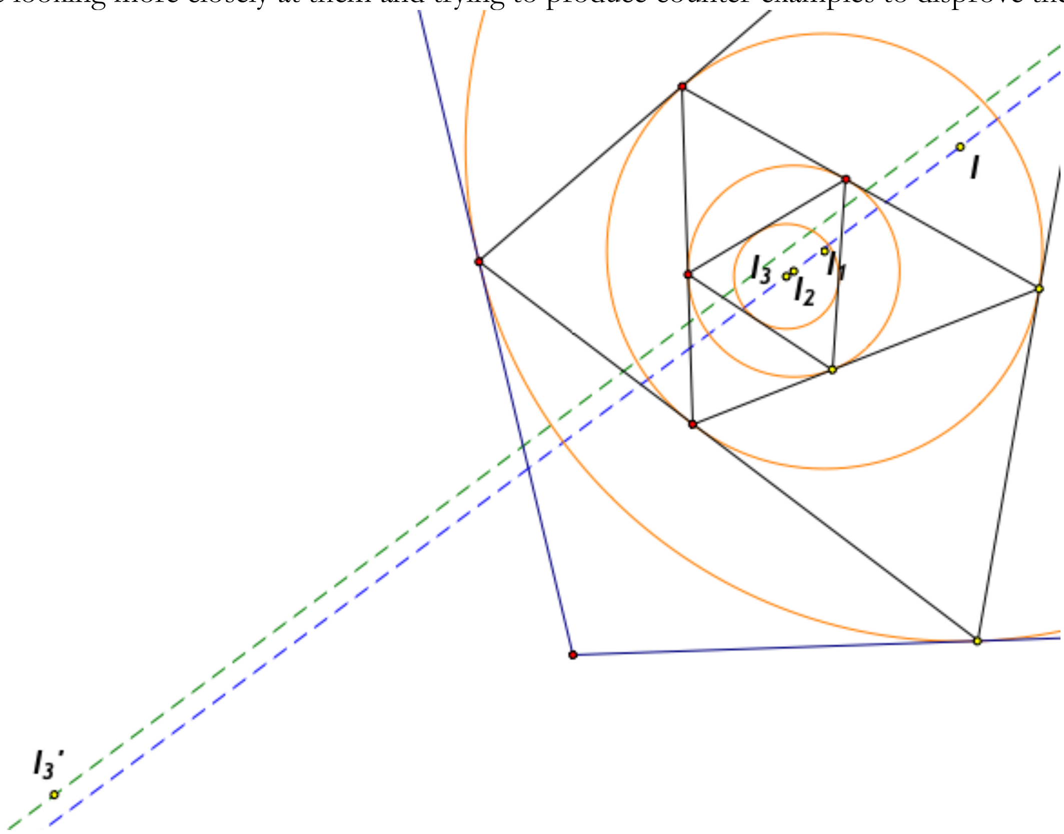

FIGURE 4: Counter-example to Conjecture 1.

Since the points were grouped so closely together we clearly needed to enlarge the figures by zooming in. This could be achieved by dragging the whole figures to make them bigger and bigger. By doing that for Conjecture 1 one could already begin to see as shown in Figure 4 that I_1 was not collinear with the other points. Alternatively, and more efficiently, one could rather use the dilation tool of the dynamic geometry software to enlarge relevant portions or elements of the figure to examine more closely.

By marking I_2 as the centre of dilation, and enlarging the blue line through I and I_3 as well as the incentres I, $I_1$and I_3, by an enlargement of 100 to 1, we noted that the line shifted to the dashed green line as shown in Figure 4. However, the images of I and I_3 still lay on the green line, whereas the image of I_1 did not. (Points I' and I'_1 were obviously off screen in the figure, but in the software one can scroll up and to the right to check where they actually lie in relation to the enlarged, green line). More over, despite I_2 appearing to lie on the constructed blue line from I to I_3, the line shift clearly showed that I_2 was not on the line. Despite our strong, initial conviction this showed conclusively that the incentres for Conjecture 1 were not collinear!

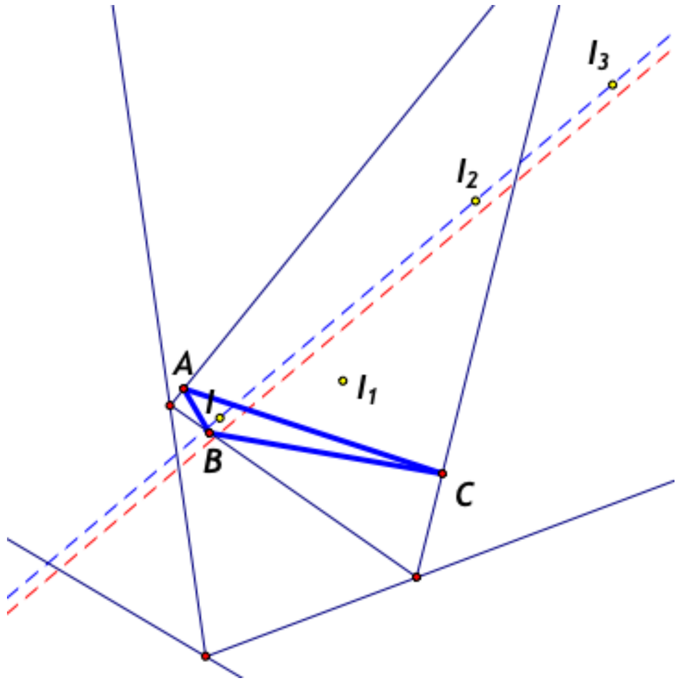

FIGURE 5: Counter-example to Conjecture 2.

Similarly we found for Conjecture 2, as shown in Figure 5, that by enlarging the upper dashed line II_3 from I_2 as centre, as well as the incentres I and I_3 by a scale factor of 100, the line shifted to the lower dashed line in red. Despite visually appearing to lie on the line and surviving the initial 'drag-test', this indicated that I_2 actually did not. (Note that again because of the large scale factor that the images I' and I'_3 were completely off-screen, but some scrolling confirmed that they were still on the enlarged red line).

Since the points lie so close to a straight line, it is important to emphasize that there is hardly any way we would have found these counter-examples by mere paper-and-pencil construction - unless we'd worked on a sheet of paper about 100 times the size of an A4 sheet, and were able to make accurate constructions using extremely large and unwieldy compasses and rulers! This episode therefore lucidly illustrates how useful computing software has become in modern day mathematical research, not only to find and formulate new conjectures, but also to enable one to disprove false statements with the production of counter-examples (compare De Villiers, 2010; Borwein, 2012).

ANOTHER REFUTATION

During our survey of related literature we also came across the following interesting result: "If the triangle formed by the feet of the angle bisectors of a triangle is iterated, the sequence of triangles converge towards an equilateral triangle" (see Trimble, 1996; Ismailescu & Jacobs, 2006). When we considered the incentres of these iterated triangles, it was difficult to see what was happening as I_2 and I_3 were so close together that they virtually lay on top of each other as shown in Figure 6. However, by enlarging I, I_1, I_2 and I_3 by a factor of 20 to 1 from the large (pink) centre in the sketch allowed us to better see where they lay in relation to each other. Clearly the incentre I was not collinear with any of the others. More over, even though I'_3 appeared to lie on the line through I'_1 and I'_2, this was not the case as an enlargement of the green line by 20 to 1 from I'_3 as centre again resulted in a shift of the line to map to the red line as shown.

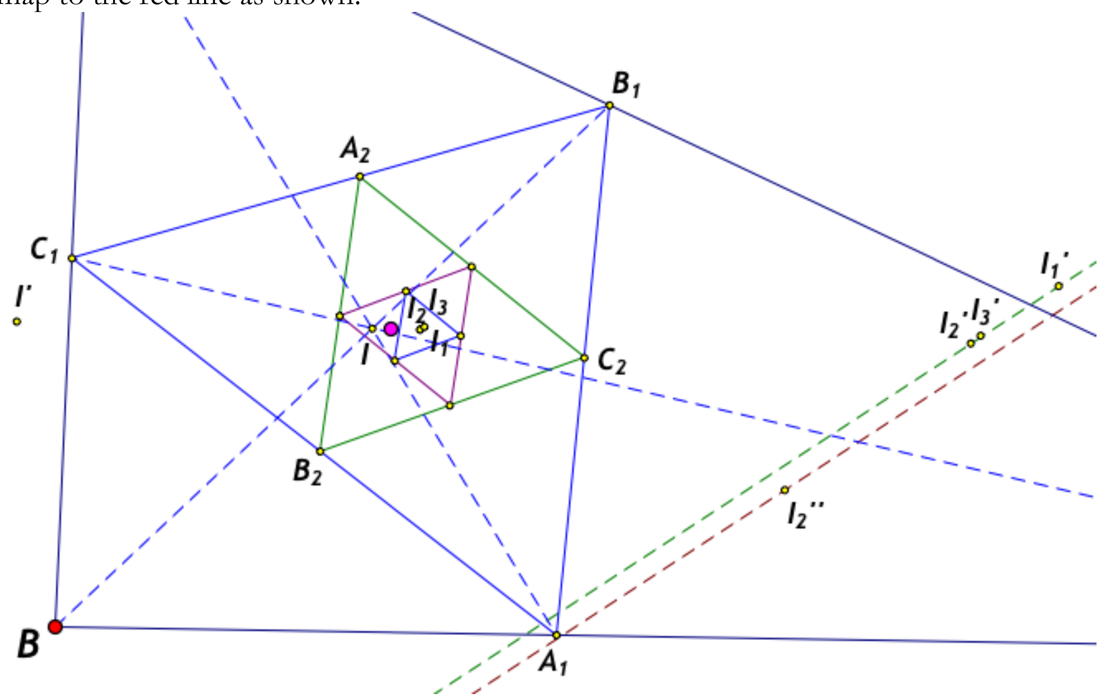

FIGURE 6: Another refutation.

CONCLUDING REMARKS

Albert Einstein the world famous physicist is reputed to have once said: "*I think and think for months and years. Ninety-nine times, the conclusion is false. The hundredth time I am right.*" In similar vein, he is often quoted as having said: "*The most important tool of the theoretical physicist is his wastebasket.*" Another popular story tells that when Albert Einstein first arrived at Princeton University in 1933, he was asked what equipment he required for his office. Apparently he replied: "*A desk, some pads and a pencil, and a large wastebasket to hold all of my mistakes.*"

Though some of these humorous anecdotes might be apocryphical, there is certainly some grain of truth in them, and does at the very least suggest some prevalence of 'mistakes' and 'errors' in Einstein's ground breaking theoretical exploration of the physical world. Similarly, research mathematicians do not all the time make only true conjectures, but many false ones as well. It is therefore just as important for a research mathematician to be able to disprove a false conjecture as it is to prove a true one. At school and undergraduate mathematics, however, textbooks in mathematics seldom give sufficient attention to cultivating 'refutation' as a critical 'habit of the mind', except perhaps with a few examples in number theory, usually rushed over in the GET phase (Grades 7-9 in South Africa), or at the start of a course in elementary number theory at university.

As strongly argued by Lakatos (1976), and illustrated historically with the Euler-Descartes theorem for polyhedra, both *local* and *global* refutation often plays an indispensable part in the development of mathematical knowledge. With global refutation is meant here the production of a counter-example that shows that a statement is false and needs to be rejected. In contrast, local or heuristic refutation typically challenges perhaps only one step in a logical argument or merely some aspect of the domain of validity of the statement; eventually leading to a more precise proof or formulation of the statement itself (and perhaps also a refinement of the concepts involved). A highly accessible example of heuristic refutation is provided in De Villiers (2003, pp. 40-44; 156-157) where learners and students are confronted with the heuristic counter-example of a quadrilateral which is dragged into the shape of a 'crossed quadrilateral', for which the interior angle sum is 720°, and not 360°.

Unfortunately textbooks (and therefore teachers also) tend to fall into the trap that Freudenthal (1973) has called the *anti-didactical inversion*; in other words, teaching only the final, polished mathematical product without showing its evolution over a period of time. By providing refined definitions, ready-made theorems and efficient algorithms, heuristic and global refutation are circumvented 'a priori', and hence cannot feature. Such a 'sanitized' approach therefore hides the adventure of 'doing real mathematics' from learners and students.

Though the investigation reported here led to only one proved conjecture and three falsified ones, it was still a useful learning experience for us. In our attempts to prove the ill-fated conjectures, we rediscovered several other known properties of incircles and excircles. We also learnt how to critically check and refute geometric conjectures using the dilation tool. As Rav (1999) has pointed out with reference to the famous Goldbach conjecture, that even if a counter-example were to be produced to it tomorrow, that would not lessen the tremendous impact of what has been learnt from various efforts to try and prove it.

It is hoped that in future more tasks and explorations in textbooks at school and university would be formulated in a more open-ended manner. For example, instead of the usual "Prove that …" rather use the more mathematically authentic version: "Explore whether the following conjecture is true or not. If true, prove it. If false, produce a counter-example."

Of course, better still would be if learners and students could be led to make their own conjectures, and then to prove or disprove them. Though that is not a feasible strategy to use all the time, they ought to at least experience a few instances like that during their mathematical education. It is also quite possible given the opportunity to pose their own problems and engage in their own explorations that learners at school (or university) could come up with original conjectures such as the one made by Clough, a Grade 11 learner in 2001 (see De Villiers, 2012).

NOTE

Prof. de Villiers and Prof. Heideman were respectively co-chair and chair of the South African Mathematics Olympiad (SAMO) committee from 1998-2014, and both are still serving on the problem solving committees.
Go to: http://www.samf.ac.za/Default2.aspx

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