An example of constructive defining:

From a GOLDEN RECTANGLE to GOLDEN QUADRILATERALS and Beyond

Part 1

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There appears to be a persistent belief in mathematical textbooks and mathematics teaching that good practice (mostly; see footnote[1]) involves first providing students with a concise definition of a concept before examples of the concept and its properties are further explored (mostly deductively, but sometimes experimentally as well). Typically, a definition is first provided as follows:

- *Parallelogram*: A parallelogram is a quadrilateral with half turn symmetry. (Please see endnotes for some comments on this definition.)
- The number $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = 2.71828\ldots$
- *Function*: A function f from a set A to a set B is a relation from A to B that satisfies the following conditions:
  (1) for each element a in A, there is an element b in B such that $<a, b>$ is in the relation;
  (2) if $<a, b>$ and $<a, c>$ are in the relation, then $b = c$.

[1]It is not being claimed here that all textbooks and teaching practices follow the approach outlined here as there are some school textbooks such as Serra (2008) that seriously attempt to actively involve students in defining and classifying triangles and quadrilaterals themselves. Also in most introductory calculus courses nowadays, for example, some graphical and numerical approaches are used before introducing a formal limit definition of differentiation as a tangent to the curve of a function or for determining its instantaneous rate of change at a particular point.

Keywords: constructive defining; golden rectangle; golden rhombus; golden parallelogram

Following such given definitions, students are usually next provided with examples and non-examples of the defined concept to 'elucidate' the definition. The problem with this overwhelmingly popular approach is that it creates the misconception that mathematics always starts with definitions, and hides from students that a particular concept can often be defined in many different equivalent ways. Moreover, students are given no idea where the definition came from and on what grounds this particular definition was chosen. By providing students with a ready-made definition, they are also denied the opportunity to engage in the process of mathematical defining themselves, and hence it unfortunately portrays to them an image of mathematics as an 'absolutist' science (Ernest, 1991).

In general, there are essentially two different ways of defining mathematical concepts, namely, *descriptive* (a posteriori) and *constructive* (a priori) defining. Descriptive definitions systematize already existing knowledge, whereas constructive definitions produce new knowledge (Freudenthal, 1973).

The purpose of this article is to heuristically illustrate the process of constructive defining in relation to a recent exploration by myself of the concept of a 'golden rectangle' and its extension to a 'golden rhombus', 'golden parallelogram', 'golden trapezium', 'golden kite', etc. Though these examples are mathematically elementary, it is hoped that their discussion will illuminate the deeper process of constructive defining.

Constructively Defining a 'Golden Rhombus'

> "... [The] *algorithmically constructive and creative definition ... models new objects out of familiar ones.*"
>
> – Hans Freudenthal (1973: 458).

Constructive (a priori) defining takes place when a given definition of a concept is changed through the exclusion, generalization, specialization, replacement or addition of properties to the definition, so that a new concept is constructed in the process.

Since there is an interesting *side-angle* duality between a rectangle (all angles equal) and a rhombus (all sides equal) (see De Villiers, 2009:55), I was recently considering how to define the concept of a 'golden rhombus'. Starting from the well-known definition of a golden rectangle as a rectangle which has its adjacent *sides* in the ratio of the golden ratio $\phi = 1.618\ldots$, I first considered the following analogous option in terms of the *angles* of the rhombus (Please see endnotes for the definition of the golden ratio):

A golden rhombus is a rhombus with adjacent *angles* in the ratio of ϕ.

Assuming the acute angle of the rhombus as x, this definition implies that:

$$\frac{180^\circ - x}{x} = \phi, \therefore x = \frac{180^\circ}{1+\phi} \approx 68.75^\circ.$$

An accurate construction of a 'golden rhombus' fulfilling this angle condition is shown in Figure 1.

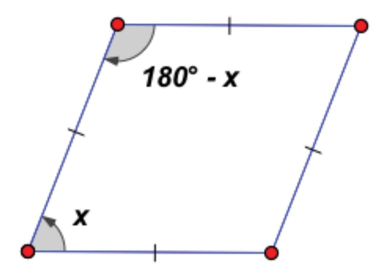

Figure 1. Golden rhombus with angles in ratio phi

Though this particular rhombus looks reasonably visually appealing, I wondered how else one might reasonably obtain or define the concept of a golden rhombus. Since a rectangle is cyclic and a rhombus has an inscribed circle, I hit upon the idea of starting with a golden rectangle *EFGH* (with $\frac{EH}{EF} = \phi$) and its circumcircle, and then constructing the rhombus *ABCD* with sides tangent to the circumcircle at the vertices of the rectangle. (Note that it follows directly from the symmetry of the rectangle *EFGH* that *ABCD* is a rhombus). Much to my surprised delight, I now found through accurate construction and measurement with dynamic geometry software as shown in Figure 2 that though the angles were no longer in the ratio phi, the diagonals for this rhombus now were!

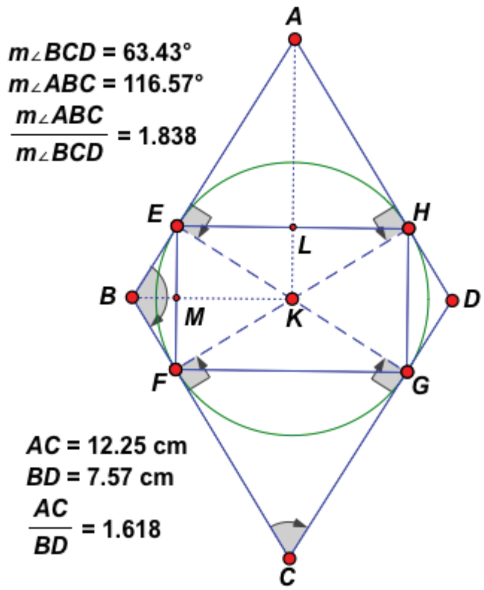

Figure 2. Golden rhombus with diagonals in ratio phi

It is not difficult to explain why (prove that) the diagonals of rhombus *ABCD* are in the ratio ϕ. Clearly triangles *ABK* and *KEM* are similar, from which follows that $\frac{AK}{BK} = \frac{KM}{EM}$. But $KM = LE$; so $\frac{AK}{BK} = \frac{LE}{EM}$. But these lengths (AK, BK); (LE, EM) are respectively half the lengths of the diagonals of the rhombus and the sides of the rectangle; hence the result follows from the property of the golden rectangle *ABCD*.

The size of the angles of the golden rhombus in Figure 2 can easily be determined using trigonometry, and the task is left to the reader. Another interesting property of both the golden rectangle and golden rhombus in this configuration is that $\tan \angle EKF = \tan \angle BCD = 2$. One way of easily establishing this is by applying the double angle tan formula, but this is also left as an exercise to the reader to verify.

Since definitions in mathematics are to some extent arbitrary, and there is no psychological reason to prefer the one to the other from a visual, aesthetic point of view (footnote[2]), we could therefore choose either one of the aforementioned possibilities as our definition. However, it seems that a better argument can be made for the second definition of a 'golden rhombus', since it shows a nice, direct connection with the golden rectangle. Also note that the second definition can be stated in either of the following equivalent forms: 1) a quadrilateral with sides constructed tangential to the circumcircle, and at the vertices, of a golden rectangle as illustrated in Figure 2; or more simply as 2) a rhombus with diagonals in the ratio of ϕ (footnote[3]).

The case for the second definition is further strengthened by the nice duality illustrated between the golden rectangle and golden rhombus in Figure 3, which shows their respective midpoint quadrilaterals (generally called '*Varignon parallelograms*'). Since the diagonals of the golden rectangle are equal, it follows that its corresponding Varignon parallelogram is a rhombus, but since its diagonals are equal to the sides of the golden rectangle, they are also in the golden ratio, and therefore the rhombus is a golden rhombus. Similarly, it follows that the Varignon parallelogram of the golden rhombus is a golden rectangle.

Constructively Defining a 'Golden Parallelogram'

Since the shape of a parallelogram with sides in the ratio of phi is variable, it seemed natural from the aforementioned to define a 'golden parallelogram' as a parallelogram *ABCD* with its sides and diagonals in the ratio phi, e.g., $\frac{AD}{AB} = \frac{BD}{AC} = \phi$ as shown in Figure 4. Experimentally dragging a dynamically constructed general parallelogram until its sides and diagonals were approximately in the golden ratio gave a measurement for $\angle ABC$ of approximately $60°$.

[2]It is often claimed that there is some inherent aesthetic preference to the golden ratio in art, architecture and nature. However, several recent psychological studies on peoples' preferred choices from a selection of differently shaped rectangles, triangles, etc., do not show any clear preference for the golden ratio over other ratios (e.g., see Grossman et al, 2009; Stieger & Swami, 2015). Such a finding is hardly surprising since it seems very unlikely that one could easily visually distinguish between a rectangle with sides in the golden ratio 1.618, or say with sides in the ratio of 1.6, 1.55 or 1.65, or even from those with sides in the ratio 1.5 or 1.7.

[3]A later search on the internet revealed that on https://en.wikipedia.org/wiki/Golden_rhombus, a golden rhombus is indeed defined in this way in terms of the ratio of its diagonals and not in terms of the ratio of its angles. A further case for the preferred choice of this definition can be also made from the viewpoint that several polyhedra have as their faces, rhombi with their diagonals in the golden ratio.

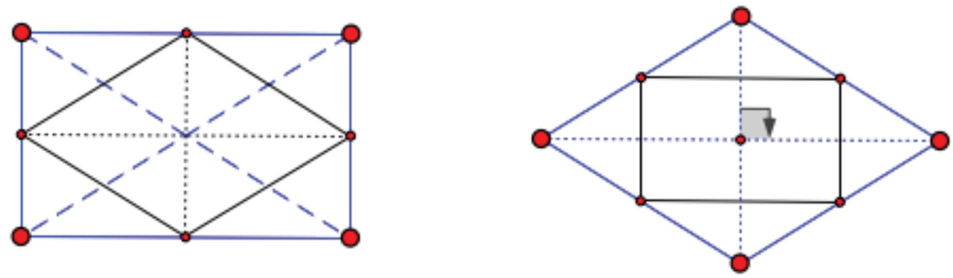

Figure 3. The Varignon parallelograms of a golden rectangle and golden rhombus

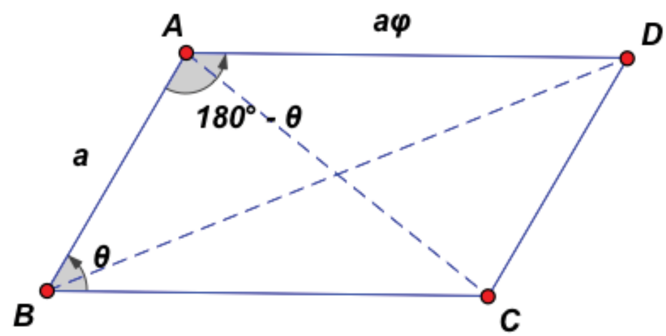

Figure 4. Golden parallelogram with sides and diagonals in the golden ratio

To prove this conjecture was not hard. Assuming $a = 1$ in Figure 4, it follows from the cosine rule that:

$$AC^2 = 1^2 + \phi^2 - 2\phi\cos\theta,$$

$$BD^2 = 1^2 + \phi^2 + 2\phi\cos\theta.$$

But since $\frac{BD}{AC} = \phi$ is given, it follows that:

$$\frac{1^2 + \phi^2 + 2\phi\cos\theta}{1^2 + \phi^2 - 2\phi\cos\theta} = \phi^2.$$

Solving this equation for $\cos\theta$ and substituting the value of ϕ gives:

$$\cos\theta = \frac{\phi^4 - 1}{2(\phi + \phi^3)} = \frac{1}{2},$$

which yields $\theta = 60^\circ$. So my experimentally found conjecture was indeed true. Accordingly, a golden parallelogram defined as a parallelogram with both its sides and diagonals in the golden ratio has 'neat' angles of 60° and 120°, and it also looks more or less visually pleasing. Equivalently, and more conveniently, we could define the golden parallelogram as a parallelogram with an acute angle of 60° and sides in the golden ratio[4] or as a parallelogram with an acute angle of 60° and diagonals in the golden ratio. That the remaining property follows from these convenient, alternative definitions is left to the interested reader to verify.

An appealing property of this golden parallelogram, consistent with that of a golden rectangle, is shown in the first two diagrams in Figure 5, namely, that respectively cutting off a rhombus at one end, or two equilateral triangles at both ends, produces another golden parallelogram. This is because in each case a parallelogram with an acute angle of 60° is obtained, and letting $a = 1$, we see that it has sides in the ratio $\frac{1}{\phi - 1}$, which is well known to equal ϕ.

In addition, constructing the Varignon parallelogram determined by the midpoints of the sides of any parallelogram as shown by the third diagram in Figure 5, it is easy to see that the sides and diagonals of the Varignon parallelogram will be in the same ratio as those of the parent

[4]Somewhat later I found that Walser (2001, p. 45) had similarly defined a golden parallelogram as a parallelogram with an acute angle of 60° and sides in the golden ratio.

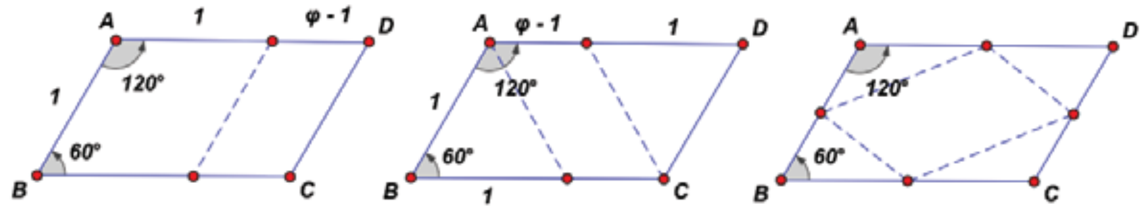

Figure 5. Construction of golden parallelograms by subdivision

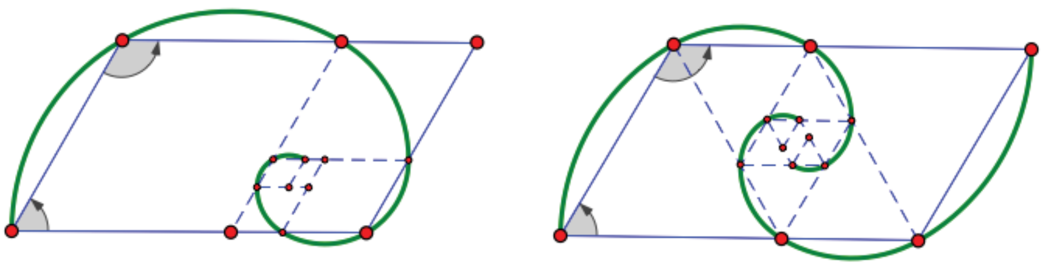
Figure 6. Spirals related to the golden parallelogram

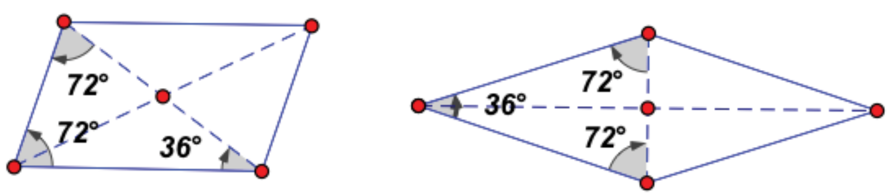

Figure 7. Alternative definitions for golden parallelogram and golden rhombus

parallelogram. Hence, the Varignon parallelogram of a parallelogram will be a golden parallelogram if and only if, the parent parallelogram is a golden parallelogram.

Of further recreational interest is that the subdividing processes of the first two diagrams in Figure 5, can be continued iteratively as shown in Figure 6, just like the golden rectangle, to produce rather pleasant looking spirals.

As was the case with the rhombus, a 'golden parallelogram' can also be constructively defined differently in terms of what is called a 'golden triangle', namely an isosceles triangle with an angle of 36° and two angles of 72° each. (It is left as an exercise to readers to verify that such a triangle has one of its legs to the base in the ratio ϕ). A golden parallelogram can therefore be obtained differently from the aforementioned by a half-turn around the midpoint of one of the legs of the golden triangle to obtain a parallelogram with sides in the ratio ϕ (see footnote[5]) as shown in the first diagram in Figure 7.

Note that using a golden triangle we can also constructively define a golden rhombus in a third way as shown in the second diagram in Figure 7. By simply reflecting a golden triangle around its 'base', we obtain a rhombus with its side to the shorter diagonal in the golden ratio. Though this 'golden rhombus' may appear too flattened out to be visually pleasing, it is of some mathematical interest as it appears in regular pentagons, regular decagons, and in combination with a regular pentagon, can create a tiling of the plane. So this is a case where visual aesthetics of a concept have to be weighed up against its mathematical relevance.

In Part-II of this article, we will explore some possible definitions for golden isosceles trapezia, golden kites, as well as a golden hexagon.

[5]Loeb & Varney (1992, pp. 53-54) define a golden parallelogram as a parallelogram with an acute angle of 72° and its sides in the golden ratio. They then proceed using the cosine rule to determine the diagonals of such a parallelogram to prove that the short diagonal is equal to the longer side of the parallelogram and hence divides it into two golden triangles.

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Endnotes

1. This is not the common textbook definition. (The usual definition is: A parallelogram is a four-sided figure for which both pairs of opposite sides are parallel to each other.) I want to emphasize that concepts can be defined differently and often more powerfully in terms of symmetry. As argued in De Villiers (2011), it is more convenient defining quadrilaterals in terms of symmetry than the standard textbook definitions. Reference: De Villiers, M. (2011). Simply Symmetric. *Mathematics Teaching*, May 2011, p34–36.

2. The Golden Ratio can be defined in different ways. The simplest one is: it is that positive number x for which $x = 1 + 1/x$; equivalently, that positive number x for which $x^2 = x + 1$. The definition implies that $x = (\sqrt{5} + 1)/2$, whose value is approximately 1.618034. A rectangle whose length : width ratio is $x : 1$ is known as a golden rectangle. It has the feature that when we remove the largest possible square from it (a 1 by 1 square), the rectangle that remains is again a golden rectangle.

3. The term Golden Rectangle has by now a standard meaning. However, terms like Golden Rhombus, Golden Parallelogram, Golden Trapezium and Golden Kite have been defined in slightly different ways by different authors.

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An example of constructive defining:

From a GOLDEN RECTANGLE to GOLDEN QUADRILATERALS and Beyond Part 2

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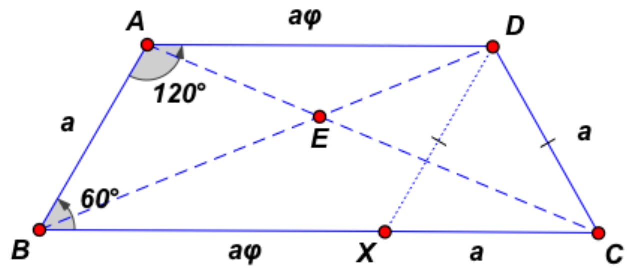
This article continues the investigation started by the author in the March 2017 issue of At Right Angles, available at: http://teachersofindia.org/en/ebook/golden-rectangle-golden-quadrilaterals-and-beyond-1 The focus of the paper is on constructively defining various golden quadrilaterals analogously to the famous golden rectangle so that they exhibit some aspects of the golden ratio phi. Constructive defining refers to the defining of new objects by modifying or extending known definitions or properties of existing objects. In the first part of the paper in De Villiers (2017), different possible definitions were proposed for the golden rectangle, golden rhombus and golden parallelogram, and they were compared in terms of their properties as well as 'visual appeal'.

In this part of the paper, we shall first look at possible definitions for a golden isosceles trapezium as well as a golden kite, and later, at a possible definition for a golden hexagon.

Constructively Defining a 'Golden Isosceles Trapezium'

How can we constructively define a 'golden' isosceles trapezium? Again, there are several possible options. It seems natural though, to first consider constructing a golden isosceles trapezium *ABCD* in two different ways from a golden parallelogram (*ABXD* in the 1st case, and *AXCD* in

Keywords: *Golden ratio, golden isosceles trapezium, golden kite, golden hexagon, golden triangle, golden rectangle, golden parallelogram, golden rhombus, constructive defining*



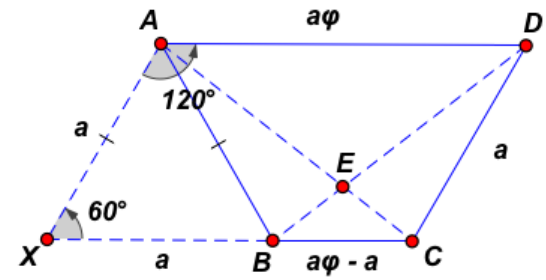


Figure 8. Constructing a golden isosceles trapezium in two ways

the 2nd case) with an acute angle of 60° as shown in Figure 8. In the first construction shown, this amounts to defining a golden isosceles trapezium as an isosceles trapezium *ABCD* with *AD // BC*, angle *ABC* = 60°, and the (shorter parallel) side *AD* and 'leg' *AB* in the golden ratio phi.

From the first construction, it follows that triangle *DXC* is equilateral, and therefore *XC* = *a*. Hence, *BC/AD* = (phi + 1)/phi, which is well known to also equal phi[1] . This result together with the similarity of isosceles triangles *AED* and *CEB*, further implies that *CE/EA* = *BE/ED* = phi. In other words, not only are the parallel sides in the golden ratio, but the diagonals also divide each other in the golden ratio. Quite nice!

In the second case, however, *AD/BC* = phi/(phi – 1) = (phi + 1) = phi squared. Also note in the second case, in contrast to the first, it is the longer parallel side *AD* that is in the golden ratio to the 'leg' *AB*, and the 'leg' *AB* is in the golden ratio with the shorter side *BC*. So the sides of this golden isosceles trapezium form a geometric progression from the shortest to the longest side, which is quite nice too!

Subdividing the golden isosceles trapezium in the first case in Figure 8, like the golden parallelogram in Figure 5, by respectively constructing a rhombus or two equilateral triangles at the ends, clearly does not produce an isosceles trapezium similar to the original. In this case the parallel sides (longest/shortest) of the obtained isosceles trapezium are also in the ratio (phi + 1), and is therefore in the shape of the second type in Figure 8. The rhombus formed by the midpoints of the sides of the first golden isosceles trapezium is also not any of the previously defined 'golden' rhombi.

With reference to the first construction, we could define the golden isosceles trapezium without any reference to the 60° angle as an isosceles trapezium *ABCD* with *AD // BC*, and *AD/AB* = phi = *BC/AD* as shown in Figure 9.

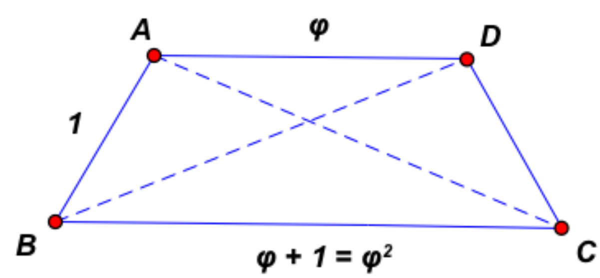


Figure 9. Alternative definition for first golden isosceles trapezium

However, this is clearly not as convenient a definition, as such a choice of definition requires again the use of the cosine formula to show that it implies that angle *ABC* = 60° (left to the reader to verify). As seen earlier, stating one of the angles and an appropriate golden ratio of sides or diagonals in the definition, substantially simplifies the deductive structure. This illustrates the important educational point that, generally, we choose our mathematical definitions for convenience and one of the criteria

[1] Keep in mind that phi is defined as the solution to the quadratic equation phi^2 – phi – 1 = 0. From this, it follows that phi = (phi + 1)/phi, phi = 1/(phi -1), phi/(phi +1) = phi + 1, or phi^2 = phi + 1.

for 'convenience' is the ease by which the other properties can be derived from it.

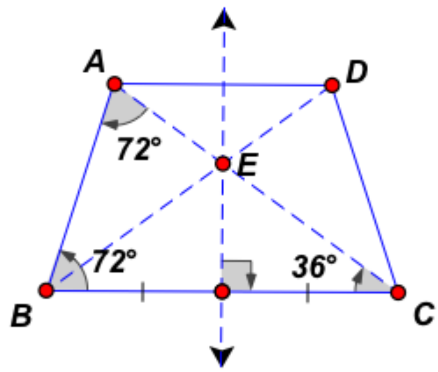


Figure 10. Third golden isosceles trapezium

Another completely different way to define and conceptualize a golden isosceles trapezium is to again use a golden triangle. As shown in Figure 10, by reflecting a golden triangle *ABC* in the perpendicular bisector of one of its 'legs' *BC*, produces a 'golden isosceles trapezium' where the ratio *BC/AB* is phi, and the acute 'base' angle is 72°. Moreover, since angle *BAD* = 108° and angle *ADB* = 36°, it follows that angle *ABD* is also 36°. Hence, *AD* = *AB* (= *DC*), and therefore the two parallel sides are also in the golden ratio, and as with the preceding case, the diagonals therefore also divide each other into the golden ratio. Of interest also, is to note that the diagonals *AC* and *DB* each respectively bisect the 'base' angles[2] at *C* and *B*.

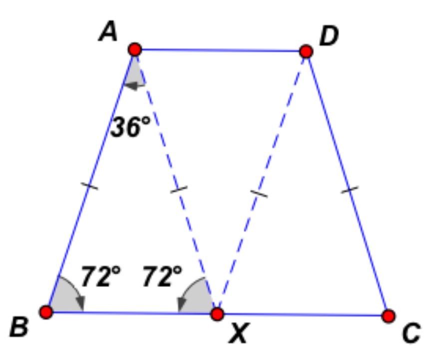


Figure 11. A fourth golden isosceles trapezium

A fourth way to define and conceptualize a golden isosceles trapezium could be to start again with a golden triangle *ABX*, but this time to translate it with the vector *BX* along its 'base' to produce a golden isosceles trapezium *ABCD* as shown in Figure 11. In this case, since the figure is made up of 3 congruent golden triangles, it follows that *AB/AD* = phi, and *BC* = 2*AD* (and therefore its diagonals also divide each other in the ratio 2 to 1).

Though one could maybe argue that the first case of a golden isosceles trapezium in Figure 8 is too 'broad' and the one in Figure 11 is too 'tall' to be visually appealing, there is little visually different between the one in Figure 10 and the second case in Figure 8. However, all four cases or types have interesting mathematical properties, and deserve to be known.

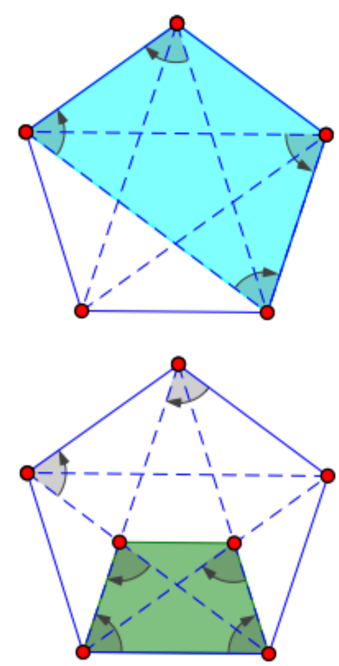


Figure 12. Golden isosceles trapezia of type 3

One more argument towards perhaps slightly favoring the golden isosceles trapezium, defined and constructed in Figure 10, might be that it appears in both the regular convex pentagon as well as the regular star pentagon as illustrated in Figure 12.

[2] In De Villiers (2009, p. 154-155; 207) a general isosceles trapezium with three adjacent sides equal is called a trilateral trapezium, and the property that a pair of adjacent, congruent angles are bisected by the diagonals is also mentioned. Also see: http://dynamicmathematicslearning.com/quad-tree-new-web.html

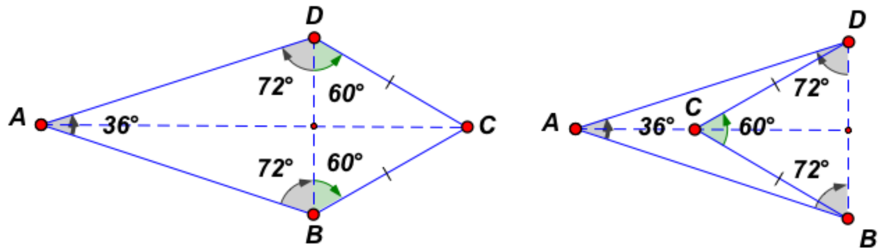


Figure 13. First case of golden kite

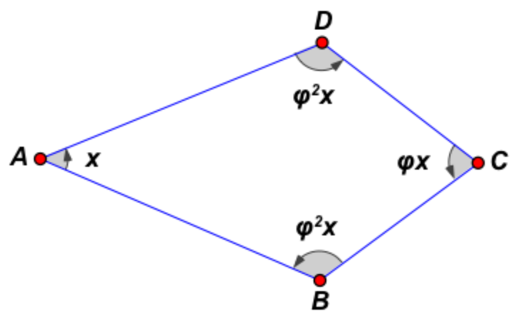


Figure 14. Second case of golden kite

Constructively Defining a 'Golden Kite'

Again there are several possible ways in which to constructively define the concept of a 'golden kite'. An easy way of constructing (and defining) one might be to again start with a golden triangle and construct an equilateral triangle on its base as shown in Figure 13. Since *AB/BD* = phi, it follows immediately that since *BD* = *BC* by construction, *AB* to *BC* is also in the golden ratio. Notice that the same construction applies to the concave case, but is probably not as 'visually pleasing' as the convex case.

Another way might be again to define the pairs of angles in the golden kite to be in the golden ratio as shown in Figure 14. Determining *x* from this geometric progression, rounded off to two decimals, gives:

$$x = \frac{360°}{1 + \phi + 2\phi^2} = 45.84°$$

Of special interest is that the angles at *B* and *D* work out to be precisely equal to 120°. This golden kite looks a little 'fatter' than the preceding convex one, and is therefore perhaps a little more visually pleasing. This observation, of course, also relates to the ratio of the diagonals, which in the first case is 2.40 (rounded off to 2 decimals) while in the case in Figure 14, it is 1.84 (rounded off to 2 decimals), and hence the latter is closer to the golden ratio phi.

To define a golden kite that is hopefully even more visually appealing than the previous two, I next thought of defining a 'golden kite' as shown in Figure 15, namely, as a (convex[3]) kite with both its sides and diagonals in the golden ratio.

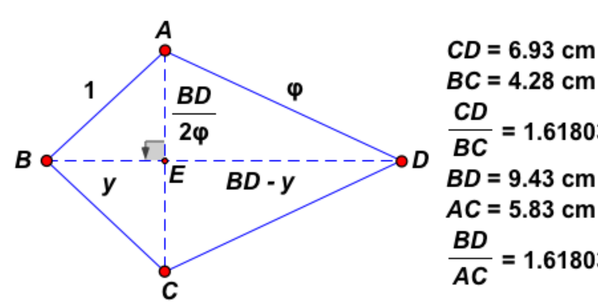


Figure 15. Third case: Golden kite with sides and diagonals in the golden ratio

[3] For the sake of brevity we shall disregard the concave case here.

Though one can drag a dynamically constructed kite in dynamic geometry with sides constructed in the golden ratio so that its diagonals are approximately also in the golden ratio, making an accurate construction required the calculation of one of the angles. At first I again tried to use the cosine rule, since it had proved effective in the case of one golden parallelogram as well as one isosceles trapezium case, but with no success. Eventually switching strategies, and assuming $AB = 1$, applying the theorem of Pythagoras to the right triangles ABE and ADE gave the following:

$$y^2 + \frac{BD^2}{4\phi^2} = 1$$

$$\frac{BD^2}{4\phi^2} + (BD - y)^2 = \phi^2.$$

Solving for y in the first equation and substituting into the second one gave the following equation in terms of BD:

$$BD^2 - 2BD\sqrt{1 - \frac{BD^2}{4\phi^2}} - \phi + 1 = 0.$$

This is a complex function involving both a quadratic function as well as a square root function of BD. To solve this equation, the easiest way as shown in Figure 16 was to use my dynamic geometry software (*Sketchpad*) to quickly graph the function and find the solution for $x = BD = 2.20$ (rounded off to 2 decimals). From there one could easily use Pythagoras to determine BE, and use the trigonometric ratios to find all the angles, giving, for example, angle $BAD = 112.28°$. So as expected, this golden kite is slightly 'fatter' and more evenly proportionate than the previous two cases. One could therefore argue that it might be visually more pleasing also.

In addition, the midpoint rectangle of the third golden kite in Figure 15, since its diagonals are in the golden ratio, is a golden rectangle.

On that note, jumping back to the previous section, this reminded me that a fifth way in which we could define a golden isosceles trapezoid might be to define it as an isosceles trapezium with its mid-segments KM and LN in the golden ratio as shown in Figure 17, since its midpoint rhombus would then be a golden rhombus (with diagonals in golden ratio). However, in general, such an isosceles trapezium is dynamic and can change shape, and we need to add a further property to fix its shape. For example, in the 1st case shown in Figure 17 we could impose the condition that BC/AD = phi, or as in the 2nd case, we can have $AB = AD = DC$ (so the base angles at B and C would respectively be bisected by the diagonals DB and AC). As can be seen, it is very difficult to visually distinguish between these two

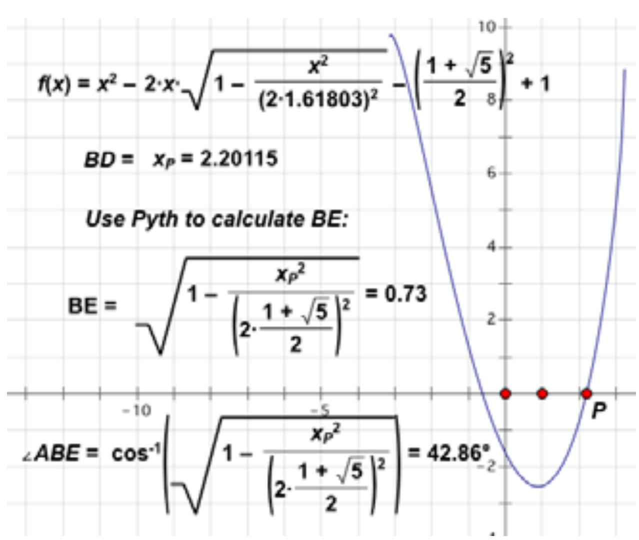


Figure 16. Solving for BD by graphing

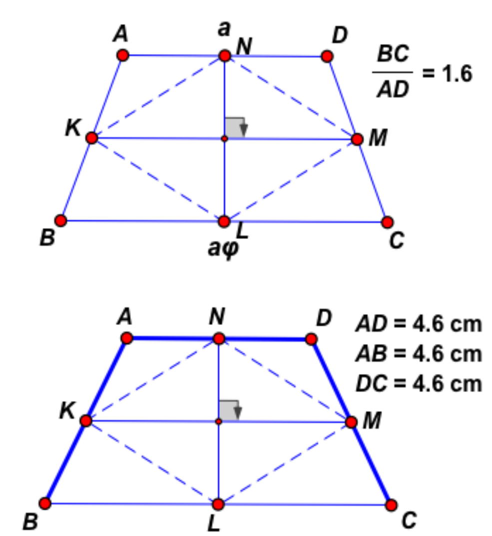


Figure 17. Fifth case: Golden isosceles trapezia via midsegments in golden ratio

cases since the angles only differ by a few degrees (as can be easily verified by calculation by the reader). Also note that for the construction in Figure 17, as we've already seen earlier, *AD* to *AB* will be in the golden ratio, if and only if, isosceles trapezium *ABCD* is a golden rectangle.

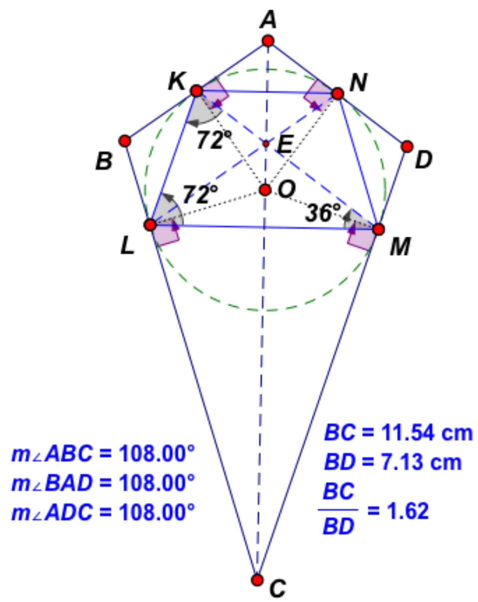


Figure 18. Constructing golden kite tangent to circumcircle of KLMN

Since all isosceles trapezia are cyclic (and all kites are circumscribed), another way to conceptualize and constructively define a 'golden kite' would be to also construct the 'dual' of each of the golden isosceles trapezia already discussed. For example, consider the golden isosceles trapezium *KLMN* defined in Figure 10, and its circumcircle as shown in Figure 18. As was the case for the golden rectangle, we can now similarly construct perpendiculars to the radii at each of the vertices to produce a corresponding dual 'golden kite' *ABCD*. It is now left to the reader to verify that *CBD* is a golden triangle (hence *BC/BD* = phi) and angle *ABC* = angle *BAD* = angle *ADC* = 108°. In addition *ABCD* has the dual property (to the angle bisection of two angles by diagonals in *KLMN*) of *K* and *N* being respective midpoints of *AB* and *AD* . The reader may also wish to verify that *AC/BD* = 1.90 (rounded off to 2 decimals), and since it is further from the golden ratio, explains the elongated, thinner shape in comparison with the golden kites in Figures 14 and 15.

Last, but not least, one can also choose to define the famous Penrose kite and dart as 'golden kites', which are illustrated in Figure 19. As can be seen, they can be obtained from a rhombus with angles of 72° and 108° by dividing the long diagonal of the rhombus in the ratio of phi so that the 'symmetrical' diagonal of the Penrose kite is in the ratio phi to the 'symmetrical' diagonal of the dart. It is left to the reader to verify that from this construction it follows that both the Penrose kite and dart have their sides in the ratio of phi. Moreover, the Penrose kites and darts can be used to tile the plane non-periodically, and the ratio of the number of kites to darts tends towards phi as the number of tiles increase (Darvas, 2007: 204). Of additional interest, is that the 'fat' rhombus formed by the Penrose kite and dart as shown in Figure 19, also non-periodically tiles with the 'thin' rhombus given earlier by the second golden rhombus in Figure 7, and the ratio of the number of 'fat' rhombi to 'thin' rhombi similarly tends towards phi as the number of tiles increase (Darvas, 2007: 202). The interested reader will find various websites on the Internet giving examples of Penrose tiles of kites and darts as well as of the mentioned rhombi.

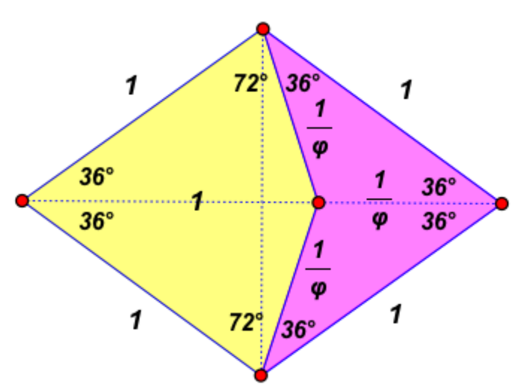


Figure 19. Penrose kite and dart

[4] In De Villiers (2009, p. 154-155; 207) , a general kite with three adjacent angles equal is called a triangular kite, and the property that a pair of adjacent, congruent sides are bisected by the tangent points of the incircle is also mentioned. The Penrose kite in Figure 19 is also an example of a triangular kite. Also see: http://dynamicmathematicslearning.com/quad-tree-new-web.html

Constructively Defining Other ‘Golden Quadrilaterals’

This investigation has already become longer than I’d initially anticipated, and it is time to finish it off before I start boring the reader. Moreover, my main objective of showing constructive defining in action has hopefully been achieved by now.

However, I’d like to point out that there are several other types of quadrilaterals for which one can similarly explore ways to define ‘golden quadrilaterals’, e.g., cyclic quadrilaterals, circumscribed quadrilaterals, trapeziums[5], bi-centric quadrilaterals, orthodiagonal quadrilaterals, equidiagonal quadrilaterals, etc.

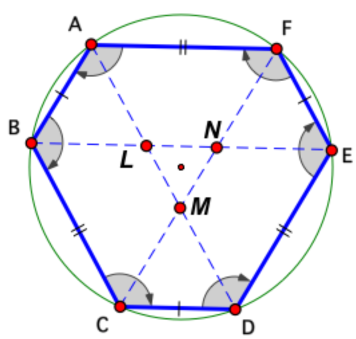


Figure 20. A golden hexagon with adjacent sides in golden ratio

Constructively Defining a ‘Golden Cyclic Hexagon’

Before closing, I’d like to briefly tease the reader with considering defining hexagonal ‘golden’ analogues for at least some of the golden quadrilaterals discussed here. For example, the analogous equivalent of a rectangle is an equi-angled, cyclic hexagon[6] as pointed out in De Villiers (2011; 2016). Hence, one possible way to construct a hexagonal analogue for the golden rectangle is to impose the condition on an equi-angled, cyclic hexagon that all the pairs of adjacent sides as shown in Figure 19 are in the golden ratio; i.e., a ‘golden (cyclic) hexagon’. It is left to the reader to verify that if *FA/AB* = phi, then *AL/LM* = phi[7], etc. In other words, the main diagonals divide each other into the golden ratio.

The observant reader would also note that *ABEF*, *ABCD* and *CDEF*, are all three golden trapezia of the type constructed and defined in the first case in Figure 8. Moreover, *ALNF*, *ABCF*, etc., are golden trapezia of the second type constructed and defined in Figure 8.

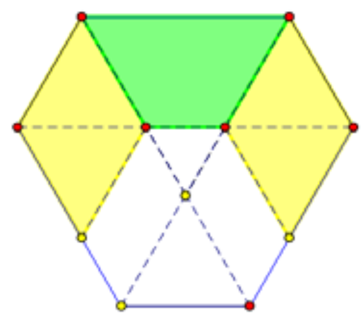


Figure 21. Cutting off two rhombi and a golden trapezium

By cutting off two rhombi and a golden isosceles trapezium as shown in Figure 21, we also obtain a similar golden cyclic hexagon. Lastly, it is also left to the reader to consider, define and investigate an analogous dual of a golden cyclic hexagon.

Concluding Remarks

Though most of the mathematical results discussed here are not novel, it is hoped that this little investigation has to some extent shown the productive process of constructive defining by illustrating how new mathematical objects can be defined and constructed from familiar definitions of known objects. In the process, several different possibilities may be explored and

[5] Olive (undated), for example, constructively defines two different, interesting types of golden trapezoids/trapeziums.

[6] This type of hexagon is also called a semi-regular angle-hexagon in the referenced papers.

[7] It was with surprised interest that in October 2016, I came upon Odom’s construction at: http://demonstrations.wolfram.com/HexagonsAndTheGoldenRatio/, which is the converse of this result. With reference to the figure, Odom’s construction involves extending the sides of the equilateral triangle LMN to construct three equilateral triangles ABL, CDM and EFN. If the extension is proportional to the golden ratio, then the outer vertices of these three triangles determine a (cyclic, equi-angled) hexagon with adjacent sides in the golden ratio.

compared in terms of the number of properties, ease of construction or of proof, and, in this particular case in relation to the golden ratio, perhaps also of visual appeal. Moreover, it was shown how some definitions of the same object might be more convenient than others in terms of the deductive derivation of other properties not contained in the definition.

The process of constructive defining also generally applies to the definition and exploration of different axiom systems in pure, mathematical research where quite often existing axiom systems are used as starting blocks which are then modified, adapted, generalized, etc., to create and explore new mathematical theories. So this little episode encapsulates at an elementary level some of the main research methodologies used by research mathematicians. In that sense, this investigation has hopefully also contributed a little bit to demystifying where definitions come from, and that they don't just pop out of the air into a mathematician's mind or suddenly magically appear in print in a school textbook.

In a classroom context, if a teacher were to ask students to suggest various possible definitions for golden quadrilaterals or golden hexagons of different types, it is likely that they would propose several of the examples discussed here, and perhaps even a few not explored here. Involving students in an activity like this would not only more realistically simulate actual mathematical research, but also provide students with a more personal sense of ownership over the mathematical content instead of being seen as something that is only the privilege of some select mathematically endowed individuals.

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