# Conway's Circle Theorem as a Special Case of a More General Side Divider Theorem

**Michael de Villiers**

***RUMEUS, University of Stellenbosch***

***profmd1@mweb.co.za***

## INTRODUCTION

The prolific and enigmatic mathematician John Conway, at the age of 82, tragically passed away on 11 April 2020 due to COVID-19 complications. He worked in the theory of finite groups, knot theory, number theory, combinatorial game theory and coding theory. He also participated in online chatrooms about mathematics, and contributed to several branches of recreational mathematics, probably the most well-known being his invention of the Game of Life.

One of the mathematical gems discovered by Conway around 2002 is the following elementary geometry theorem: Given any triangle $ABC$, extend its sides at each vertex by the length of the side opposite each vertex as shown in Figure 1. Then $PQRSTU$ is cyclic.

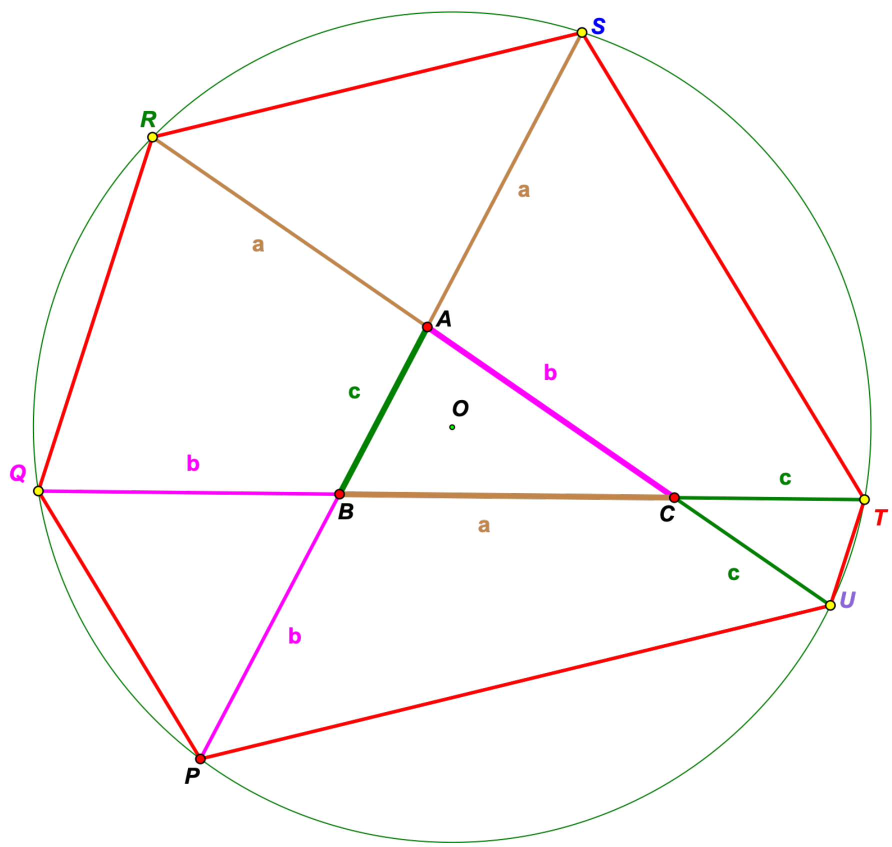


FIGURE 1

## SIDE DIVIDER THEOREM

As it turns out, this result is merely a special case of a more general 'side divider' theorem proved in De Villiers (1994, 2007), which can be formulated as follows:

> Given any $\Delta ABC$ with an arbitrary point $P$ on line $AB$, construct $BQ = BP$, $CR = CQ$, $AS = AR$, $BT = BS$, and $CU = CT$. Then $AU = AP$, and $PQRSTU$ is cyclic (see Figure 2).

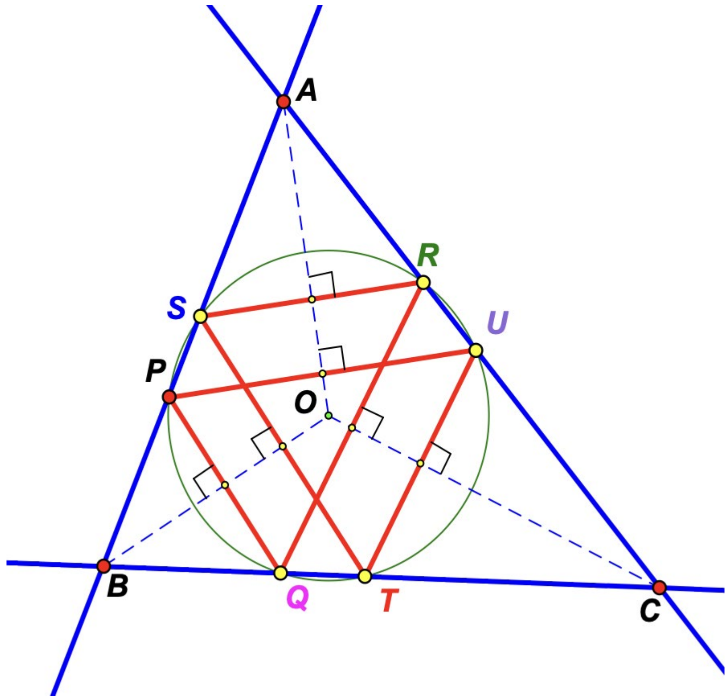


**FIGURE 2**

A dynamic geometry sketch illustrating this theorem is available online at the link below, and readers are encouraged to engage with this interactively before reading further.

http://dynamicmathematicslearning.com/conway-circle-as-special-side-divider-theorem.html

The proof of the 'side divider' theorem is quite straight forward and should readily be accessible to high school learners at different levels.

## PROOF

Label the sides $AB$, $BC$ and $CA$ respectively as $c$, $a$ and $b$. Let $AP = x$, then:

- $BQ = BP = c - x$
- $CR = CQ = a - (c - x) = a - c + x$
- $AS = AR = b - (a - c + x) = b - a + c - x$
- $BT = BS = c - (b - a + c - x) = a - b + x$
- $CU = CT = a - (a - b + x) = b - x$

Hence: $AU = b - (b - x) = x = AP$

Since $PQB$ and $STB$ are isosceles triangles sharing sides and a common vertex $B$, it follows that their axes of symmetry coincide. In other words, the perpendicular bisector of both $PQ$ and $ST$ coincides with the angle bisector of $\hat{B}$. Similarly, the perpendicular bisectors of the other two pairs of isosceles triangles, $UTC$ and $RQC$, and $RSA$ and $UPA$, respectively coincide with the angle bisectors of $\hat{C}$ and $\hat{A}$. Therefore, the perpendicular bisectors of all the sides of $PQRSTU$ are concurrent at the incentre of $\Delta ABC$, and hence the incentre is equidistant from the vertices of $PQRSTU$, which implies that it is cyclic, and completes the proof.

This proof is completely general (using directed distances), and the point $P$ can be chosen anywhere on line $AB$, even outside segment $AB$ (i.e. on either extension outside). If $x$ is negative, then $AP$ lies in the opposite direction of the representation in Figure 2, and will lie completely outside segment $AB$. Since the perpendicular bisectors of the sides remain concurrent at the incentre of $\Delta ABC$, all circles $PQRSTU$ remain concentric with the incircle of $\Delta ABC$.

Note that in the special case when $P$ is chosen at the tangential point of the incircle of $\Delta ABC$, the points $P$ and $S$ coincide, and the circle $PQRSTU$ reduces to the incircle. The hexagon $PQRSTU$ obviously has opposite sides parallel ($PQ \parallel ST$, $RS \parallel UP$, $TU \parallel QR$) as a result of the three pairs of isosceles triangles, each pair sharing a vertex and sides. Also note that the hexagon degenerates to a (crossed) isosceles trapezium when $P$ is placed at either one of vertices $A$ or $B$.

**CONWAY'S CIRCLE THEOREM AS A SPECIAL CASE**

Conway's Circle Theorem (as described at the beginning of the article and illustrated in Figure 1) is now easily obtained as a special case of the Side Divider Theorem. In Figure 2 simply choose $P$ (or drag it in a dynamic sketch) outside $AB$, i.e. on the extension of $AB$, so that $BP = b$. This choice of $P$ produces precisely Conway's configuration as shown in Figure 1, giving us an immediate visual proof of the theorem since the conditions of the Side Divider Theorem are clearly met, namely $BQ = BP$, $CR = CQ$, $AS = AR$, $BT = BS$ and $CU = CT$. Note that even though the three pairs of isosceles triangles are now on opposite sides of the vertices of $\Delta ABC$, each pair still shares the same opposite angle and sides as before. Therefore, the perpendicular bisectors of all the sides of $PQRSTU$ remain concurrent at the incentre of $\Delta ABC$, and $PQRSTU$ remains cyclic.

**ANGLE DIVIDER THEOREM**

Although the 'side-divider' theorem (and its special case of Conway's Circle Theorem) is quite remarkable in itself, for me personally it obtains additional beauty in having a lovely 'side-angle' dual as discussed in De Villiers (1994, 2007). This dual for any $\Delta ABC$ can be formulated as follows, and is illustrated in Figure 3: Construct any angle divider (line) $AX$ of $\hat{A}$. Rotate line $AB$ around $B$ by the (directed) $X\hat{A}B$, and label its intersection with $AX$ as $P$. Note that this construction obviously implies that $A\hat{B}P = P\hat{A}B$. In a similar way, construct angle divider $CQ$ of $\hat{C}$ so that $B\hat{C}Q = Q\hat{B}C$, angle divider $AR$ of $\hat{A}$ so that $C\hat{A}R = R\hat{C}A$, angle divider $BS$ of $\hat{B}$ so that $A\hat{B}S = S\hat{A}B$, and angle divider $CT$ of $\hat{C}$ so that $B\hat{C}T = T\hat{B}C$. If $U$ is the intersection of lines $CT$ and $AP$, then $U\hat{C}A = C\hat{A}U$, and $PQRSTU$ is a tangential (circumscribed) hexagon.

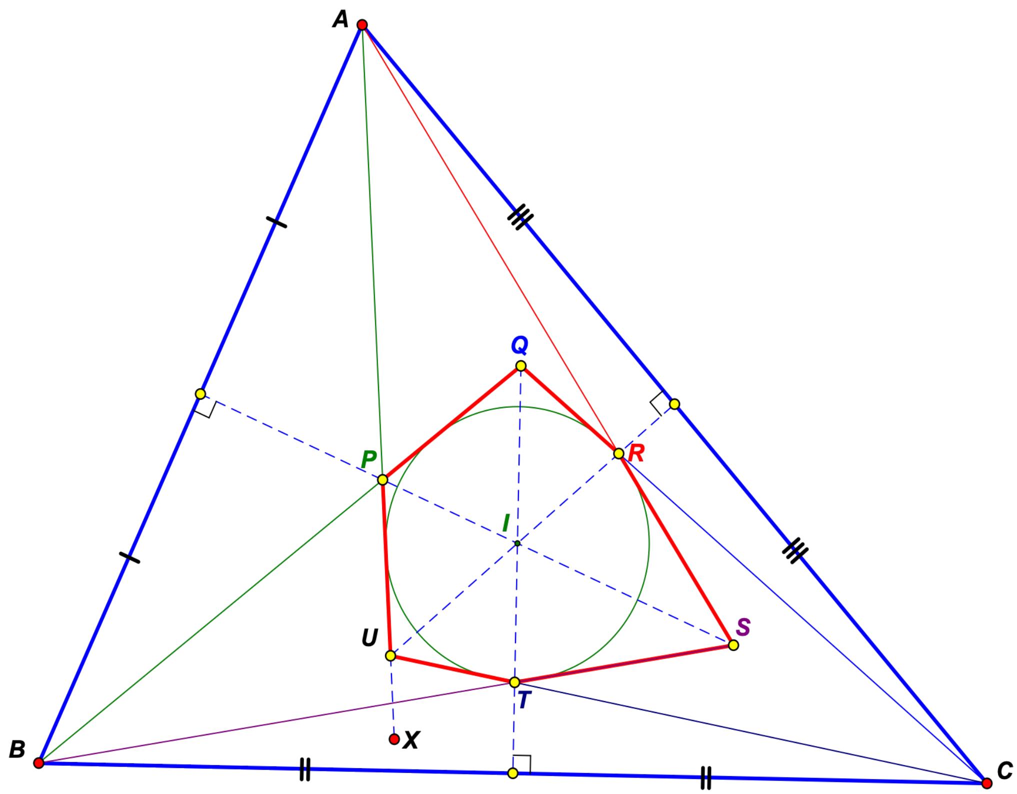


**FIGURE 3**

The proof that $U\hat{C}A = C\hat{A}U$ is straight forward and similar to the one above, and is left to the reader to verify. Here the isosceles triangles $ABP$ and $ABS$ have the same axis of symmetry, namely the perpendicular bisector of $AB$, which is therefore also the common angle bisector of $A\hat{P}B$ $(= Q\hat{P}U)$ and $A\hat{S}B$. The same holds for the other two pairs of isosceles triangles on $BC$ and $CA$ respectively. Hence, since the perpendicular bisectors of $\Delta ABC$ are concurrent, this implies that the angle bisectors of the angles of $PQRSTU$ are also concurrent at the circumcentre of $\Delta ABC$. Therefore, the circumcentre is equidistant from all the sides of $PQRSTU$ (or extended sides), and it has an incircle concentric with the circumcircle of $\Delta ABC$.

Again the result and proof is perfectly general (using directed angles), and the angle divider $AX$ may also fall outside $\hat{A}$. The reader may also wish to explore the dynamic construction illustrating this theorem, which is available at the same URL provided earlier. The reader is also encouraged to use it to consider special cases.

**CONCLUDING COMMENTS**

It is not known to me whether Conway was aware of the more general 'side-divider' theorem presented above when he posed his problem in 2002. However, it is likely that he knew of it and precisely posed the curious, special case in the hope that in proving it, others might discover its further generalization.

Both the 'side-divider' theorem, and its dual, the 'angle-divider' theorem, respectively generalize to tangential (circumscribed) and cyclic polygons, and are discussed further in my book (De Villiers, 1994, 2007). At the 1996 AMESA Congress, in a paper about the 'side-angle' duality in geometry, I mentioned among others, the quadrilateral cases for both results (De Villiers, 1996). In NCTM Conferences in 1999 and 2000, respectively in San Francisco and Chicago, I may also have mentioned these theorems in my presented papers. In April 2001, my paper at the Annual Meeting of the Mathematical Association at St. Martin's College specifically dealt in some detail with both theorems and their generalizations, using the concept of directed distances and angles. Personally I first became aware of the 'side-divider' theorem in a short paper by Van Duyn (1987) who discussed the case for a tangential (circumscribed) quadrilateral. This latter paper inspired me to investigate & generalize it further to the general 'side-divider' theorem for tangential polygons.

Conway's Circle Theorem has apparently appeared on Math Camp T-shirts for high school students in the USA, and got some more attention shortly after his death in April 2020 where it was mentioned in blogs by Matt Baker (2020), Colin Beveridge (2020) and Colm Mulcahy (2020), where some other elementary proofs are also given. So far the only paper that I've been able to find that not only generalizes Conway's Circle Theorem to the 'side-divider' theorem for a triangle, but also generalizes it to tangential polygons, is that of Braude (2021). However, Braude's generalizations are unfortunately restricted to only the extensions of the sides outside the polygons (apparently not considering/mentioning, with reference to Figures 1 and 2, that $P$ could fall on (inside) side $AB$).

While the 'angle-divider' theorem and its generalization to cyclic polygons seems less known, the 'side-divider' theorem (at least for a triangle) has been around for some time, and might have been well known much longer in problem solving circles in other parts of the world, e.g. Eastern Europe and Asia. It has often been used in Olympiad training of talented students locally in South Africa as well as appearing from time to time in student journals. In fact, the first part of the 'side-divider' theorem featured as a question in the March 1999 Sharp Calculator Competition of the *Mathematical Digest*, published by the University of Cape Town. This question was presumably inspired by David Gale (1998) from the University of California, Berkeley, who wrote the following little amusing limerick about the first part of the theorem:

**Euclid's Last (or Lost) Theorem**

In a triangle called *ABC*
Pick a point on *AB*, call it *P*. Pick a *Q* on *BC*,
Where *BQ* is *BP*.
Ah the joys of pure geo-me-tree!
On *CA* pick an *R*, oh please do, Where *CR* is exactly *CQ*, And now pick an *S*
On *AB*, more or less,
So that "*AS* is *AR*" is true.
On *BC* the next letter is *T*,
Where *BT* is *BS*, don't you see.
On *CA* pick a *U*,
And you'll know what to do,
Next what's this? We've arrived back at *P*!
Now some proofs were soon found close at hand,
But it didn't turn out quite as planned,
For though not very large
(They would fit in the margin) regrettably, none of them scanned.

Not able to resist the temptation, I'd like to suggest adding the following two lines to Gale's limerick to cover the second part of the 'side-divider' theorem for a triangle:

Lastly, tighten your girdle,
because *PQRSTU* lies on a circle.

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