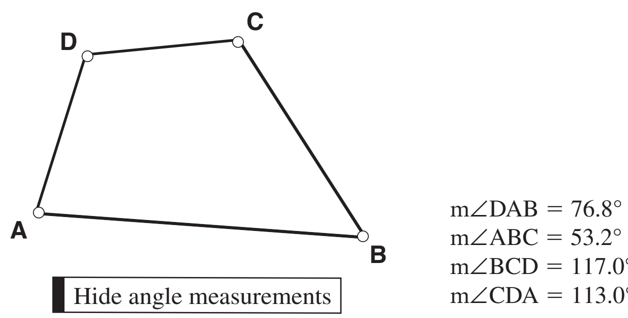
You may have already made a conjecture about the sum of the angles in a convex or concave quadrilateral. You may have also explained why the conjecture is true. Does that conjecture apply to quadrilaterals that cross themselves? In this activity, you'll investigate that question.

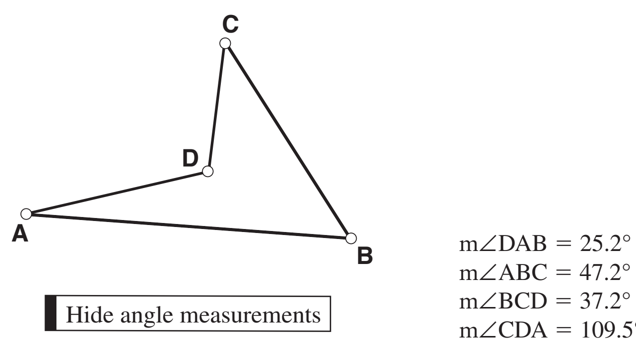
CONJECTURE

- Open the sketch **Crossed Quad Sum.gsp.** This sketch shows a convex quadrilateral and its angle measures.
- Drag any vertex and observe the angle measures and sum. For now, keep the quadrilateral convex.



1. Write down what you observe about the sum of the measures of the interior angles in a convex quadrilateral.

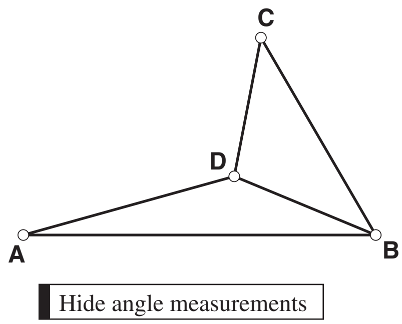
- Drag a vertex so that the quadrilateral is concave and observe the angle measures and sum as you drag.



2. What do you observe about the angle sum in concave quadrilaterals? Is that what you expected to happen? Why or why not?

3. In your sketch, draw the diagonal of the concave quadrilateral. If you've explained why the quadrilateral angle sum conjecture is true for convex quadrilaterals, think about that explanation. In what way or ways does your figure now contradict the measures and angle sum Sketchpad is displaying?

To make the diagonal dashed, select it, then choose **Line Width: Dashed** from the Display menu.



m∠DAB = 15.4°
m∠ABC = 59.5°
m∠BCD = 41.5°
m∠CDA = 116.3°
m∠DAB + m∠ABC + m∠BCD + m∠CDA = 232.7°

Confused? You should be! If you understand the explanation as to why the sum of the measures of the interior angles of a quadrilateral is 360°, you should see that it should apply to concave as well as convex quadrilaterals. Yet Sketchpad does not report the sum as 360° in concave quadrilaterals. What's going on? Is the explanation wrong? Is Sketchpad broken? In the next questions and steps, you'll discover what's going on and how to remedy the problem.

4. Look at the figure in Question 3. Three of the four interior angle measures are correct, but one of the measures is not of an interior angle. This is because Sketchpad always shows angle measures less than 180°. Identify the angle in the figure whose measure is greater than 180°, calculate its measure, and write it below. Does using this measure give you 360° for the sum of the interior angles in the figure?

An angle with measure greater than 180°, like the angle you identified in Question 4, is called a *reflex angle.* As you've discovered, Sketchpad doesn't measure reflex angles. However, Sketchpad does measure arc angles, and arc angle measures can be greater than 180°. Arcs will solve our problem of measuring reflex angles.

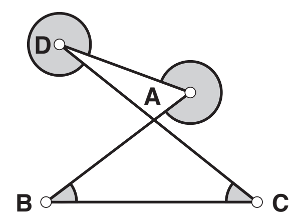
- Press the *Show arc angles* button.
- Drag a vertex of the quadrilateral and observe the arc angle measures and their sum.

5. Which measures—the simple angle measures or the arc angle measures—do you think are more useful for investigating the interior angles of a general (convex or concave) quadrilateral? Why?

- Select the diagonal and delete it.

6. Before dragging any farther, what do you think will happen to the sum of the measures of the interior angles if two sides of quadrilateral *ABCD* are crossed?

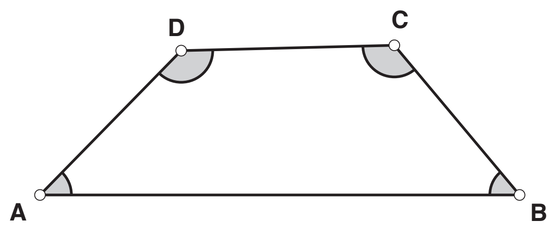
- Now drag a vertex of your quadrilateral until it becomes a crossed quadrilateral. Observe what happens to the arcs and to the measures as you drag a vertex around.



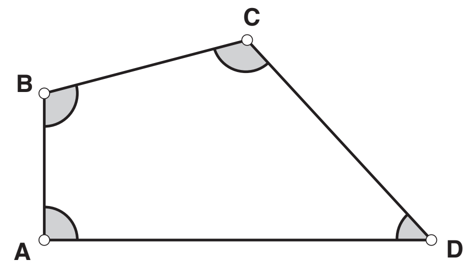
7. You should observe that two of the arcs in any crossed quadrilateral are always reflex angles. Does it make sense to call these reflex angles "interior" angles? Why or why not?

"Yes" and "no" are both acceptable answers to Question 7. You may object to calling the reflex angles "interior" angles because they seem to fall outside the polygon. On the other hand, when a polygon crosses itself, it's no longer obvious what the outside or the inside is. It's possible to define the interior angles of a crossed quadrilateral in a way that's consistent with non-crossed quadrilaterals.

Imagine you're walking around the first quadrilateral shown on the next page, alphabetically, from *A* to *B* to *C* and so on. The interior of the quadrilateral is always to your left. Now imagine you're walking around the second quadrilateral, from *A* to *B* to *C* and so on. This quadrilateral has a different orientation, but it's still clear where the interior is: It's to your right. So as you walk around, the interior angles of a quadrilateral can all be either on the right or on the left.

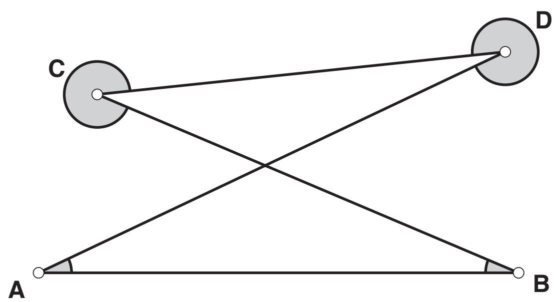


The interior is to your left as you walk from *A* to *B* to *C* and so on.

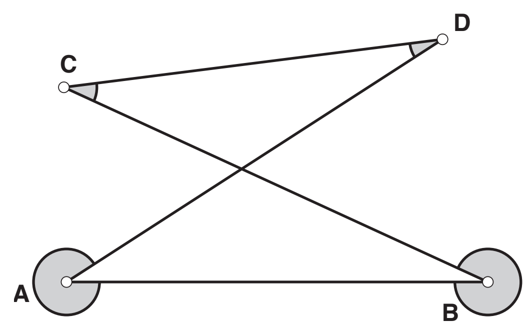


The interior is to your right as you walk from *A* to *B* to *C* and so on.

In a crossed quadrilateral, like either of those shown below, it's arbitrary whether the interior is to your right or your left as you walk around. So for the purpose of defining the interior angles, stay with what you know about non-crossed quadrilaterals: The angles are all on the right as you walk around, or they're all on the left. In a crossed quadrilateral, you can call either set the "interior" angles.



The set of interior angles is to your left as you walk from *A* to *B* to *C* and so on.

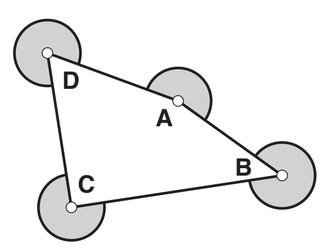


The set of interior angles is to your right as you walk from *A* to *B* to *C* and so on.

Drag a vertex to observe various different crossed quadrilaterals.

8. According to the definition of interior angles given above, what is the sum of the measures of the interior angles of a crossed quadrilateral?

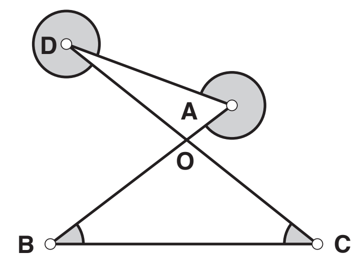
9. You can keep dragging until you cross another pair of sides. (In a sense, you are turning the quadrilateral inside out.) Describe your results. Which correctly reports the interior angle sum: the simple angle measures, the arc angle measures, or neither? Explain.



CHALLENGE Explain your observations in Question 8. Can you derive a general formula for the interior angle sum of any given convex, concave, or crossed polygon?

EXPLAINING

Work through the steps below to explain logically why the measures of the interior angles of a crossed quadrilateral add up to 720°.



10. Express the measures of reflex angles *ADC* and *BAD* respectively in terms of the measures of acute angles *ADC* and *BAD*.

11. Express the measure of angle *BOD* in terms of the measures of acute angles *ADC* and *BAD*. Explain your expression.

12. Express the measure of angle *BOD* in terms of the measures of angles *BCD* and *ABC*. Explain your expression.

13. From Questions 11 and 12, what can you now conclude about the relationship between the sum of the measures of acute angles *ADC* and *BAD*, and about the sum of the measures of angles *BCD* and *ABC*?

14. From Question 13, what can you now conclude about the sum of the measures of reflex angle *ADC*, reflex angle *BAD*, angle *BCD*, and angle *ABC*?

Present Your Explanation

Write out your explanation for presentation to the class or to your group. Your summary may be in paper form or electronic form and may include a presentation sketch in Sketchpad. You may want to discuss the summary with your partner or your group.

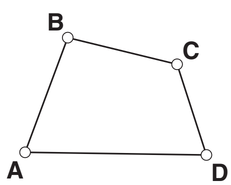
CROSSED QUADRILATERAL SUM (PAGE 40)

Sketchpad makes it natural for students to explore shapes that are not traditionally studied. These shapes include concave and crossed polygons. One purpose of this activity is to make students aware of the limitations of the standard angle measurement of Sketchpad and to let them discover that (and explain why) the sum of the measures of the interior angles of a crossed quadrilateral is 720°.

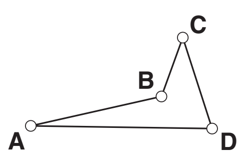
For a concave quadrilateral, the measures of the four interior angles actually do not sum to 360° in Sketchpad. The reason for this is that Sketchpad does not measure angles greater than 180°. As the activity demonstrates, this can be rectified by measuring the corresponding arc lengths.

This activity begins with a lot of reading that might be most efficiently presented to students as a whole-class demonstration as follows:

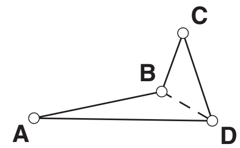
$m\angle DAB = 70.4^\circ$
$m\angle ABC = 96.1^\circ$
$m\angle BCD = 122.2^\circ$
$m\angle CDA = 71.4^\circ$
$m\angle DAB + m\angle ABC + m\angle BCD \; m\angle CDA = 360^\circ$



$m\angle DAB = 13.2^\circ$
$m\angle ABC = 123.9^\circ$
$m\angle BCD = 39.4^\circ$
$m\angle CDA = 71.4^\circ$
$m\angle DAB + m\angle ABC + m\angle BCD + m\angle CDA = 247.9^\circ$



$m\angle DAB = 13.2^\circ$
$m\angle ABC = 123.9^\circ$
$m\angle BCD = 39.4^\circ$
$m\angle CDA = 71.4^\circ$
$m\angle DAB + m\angle ABC + m\angle BCD + m\angle CDA = 247.9^\circ$



Construct a convex quadrilateral. Measure the angles and sum them. Drag a vertex until the quadrilateral becomes concave. The sum will start to vary at this point. Ask students whether they think this means the quadrilateral angle sum theorem applies only to convex quadrilaterals. Assuming some do reach this erroneous conclusion, draw a diagonal and ask students to recall the explanation from the Quadrilateral Angle Sum activity. Is there any reason the explanation for the quadrilateral angle sum shouldn't apply to this concave quadrilateral? At this point, students should begin to notice that Sketchpad is not measuring the interior angle whose measure is greater than 180° but is instead measuring an angle outside the polygon. Explain that Sketchpad always measures angles less than 180°, but that students will work in a sketch that gets around that limitation by measuring arc angles instead. Then briefly direct students' attention to the figures on the worksheet that illustrate a possible definition of "interior" angles in a crossed quadrilateral. Students will then be prepared to work on their own starting at the Conjecture section of the activity.

Prerequisites: Students should be familiar with the sum of the measures of the angles of a convex quadrilateral. It is even better preparation for them to have completed the previous activity, Quadrilateral Angle Sum. It will also help if they know that the sum of the measures of the exterior angles of a simple closed polygon is 360°.

Sketch: Crossed Quad Sum.gsp.

CONJECTURE

1. The sum of the measures of the interior angles of a convex quadrilateral is 360°.
2. The standard angle measures do not sum to 360° for a concave quadrilateral; the sum actually varies.
3. Responses will vary. The diagonal *DB* divides the quadrilateral into two triangles, so the sum of the measures of the interior angles should still be $2 \times 180° = 360°$.
4. Measure of interior angle $CDA = 360° -$ standard Sketchpad measure of angle *CDA*.
5. The sum of the arc angle measures gives the correct value for the interior angle sum of both a convex and a concave quadrilateral.
6. Answers will vary. Students might guess that the sum of the measures of the interior angles of a crossed quadrilateral is also 360°.
7. Responses will vary.
8. The sum of the measures of the interior angles of a crossed quadrilateral is 720°, as shown by the arc angle sum.

It should also be pointed out that some students may at first want to argue that crossed quadrilaterals are not quadrilaterals at all. This view corresponds to the technique of *monster-barring* described by Imre Lakatos in his famous book *Proof and Refutations* (1976). This can create a valuable opportunity for some classroom discussion and debate. Essentially, the issue is how one chooses to define the quadrilaterals, or for that matter, polygons in general. However, within a dynamic geometry environment, a simple closed polygon is easily transformed into a crossed polygon, and it therefore seems natural to simply consider the crossed polygons as special cases.

9. Neither. The sum of the measures of the arc angles is now displayed as 1080°. There are now at least three reflexive arc angles, so the arc angle sum has to be more than that in the crossed case. However, since all the arc angles fall outside in this case, they can hardly be considered "interior" angles. The actual sum of the measures of the "interior" angles can be determined from the arc angles as follows:

$$(360° - m\angle A) + (360° - m\angle B) + (360° - m\angle C) + (360° - m\angle D) = 1080° \longrightarrow$$

$$m\angle A + m\angle B + m\angle C + m\angle D = 4 \times 360° - 1080° = 360°$$

The sum of the simple angle measures will be correct only if the "inside out" quadrilateral is convex. If it is concave, we have the same problem as before: Sketchpad does not display the measurement of the one reflex angle correctly, so the angles do not sum to 360°.

EXPLAINING

10. measure of reflexive $\angle ADC = 360° -$ measure acute $\angle ADC$ and measure of reflexive $\angle BAD = 360° -$ measure acute $\angle BAD$.
11. $m\angle BOD = m$ acute $\angle ADC + m$ acute $\angle BAD$ (exterior angle of ΔDOA).
12. $m\angle BOD = m\angle BCD + m\angle ABC$ (exterior angle of ΔBOC).
13. m acute $\angle ADC + m$ acute $\angle BAD = m\angle BCD + m\angle ABC$.
14. m reflexive $\angle ADC + m$ reflexive $\angle BAD + m\angle BCD + m\angle ABC = (360° - m$ acute $\angle ADC) + (360° - m$ acute $\angle BAD) + (m$ acute $\angle ACD) + (m$ acute $\angle BAD) = 720°$.

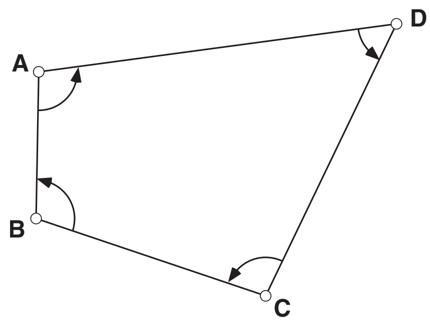
ADDITIONAL NOTES

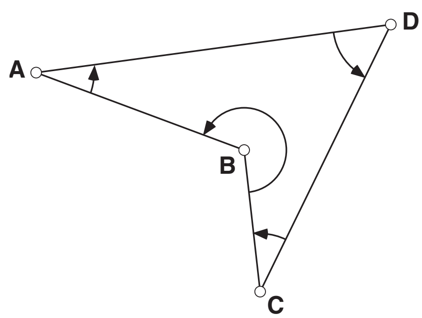
Some students may have difficulty understanding why two reflexive angles that seem to fall outside a crossed quadrilateral are considered "interior" angles. A more detailed discussion, such as the following, may help.

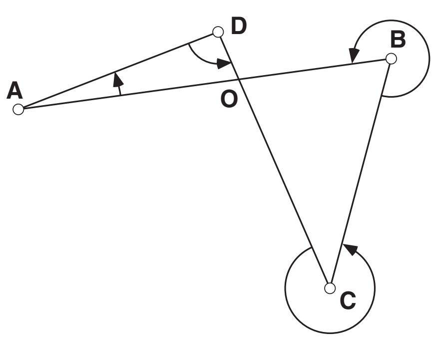
One way to extend the notion of interior angles to crossed quadrilaterals is by first carefully analyzing and defining the notion of internal angles for convex and concave quadrilaterals and then consistently applying that definition to crossed quadrilaterals. (This is a strategy often used in mathematics to extend certain concepts beyond their original domain(s)—for example, in extending positive integers to negative integers.)

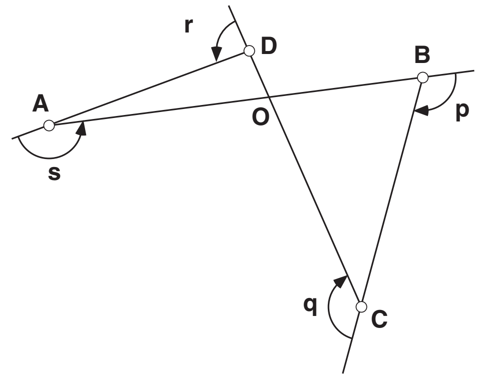
Suppose we walk counterclockwise from *A* to *B*, *B* to *C*, and so on, around the perimeters of the convex and concave quadrilaterals shown in the figure. The internal angle at each vertex can then be defined precisely as the angle through which the *next side* must be rotated *counterclockwise* (with the vertex as rotation center) to coincide with the *preceding side.*

Using exactly this definition, the internal angles of a crossed quadrilateral can now be obtained by walking around its perimeter as shown.









The sum of the measures of the interior angles of a crossed quadrilateral can also be determined from the sum of the measures of the exterior angles (see the figure above). Here the sum of the measures of the turning (exterior) angles is equal to 0°. Imagine yourself as a turtle walking around the perimeter, turning at each vertex. After turning clockwise twice, you then turn counterclockwise twice to arrive back at *A* facing in the same direction as you were at the beginning. Therefore, $m\angle p + m\angle q + m\angle r + m\angle s = 0°$, and the interior angle sum measure is given by

$$(180° - m\angle p) + (180° - m\angle q) + (180° - m\angle r) + (180° - m\angle s)$$

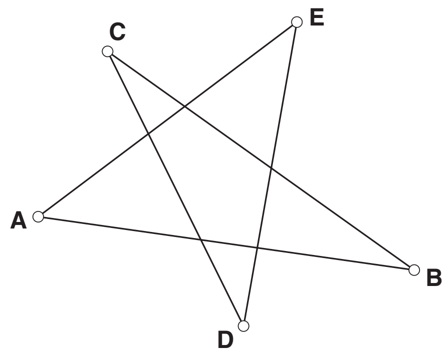
$$= 4 \times 180° - (m\angle p + m\angle q + m\angle r + m\angle s)$$

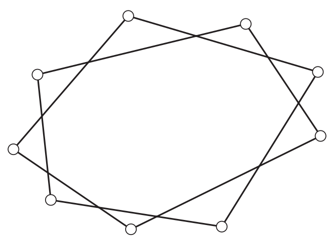
$$= 720° - 0° = 720°$$

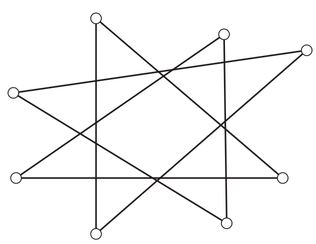
The traditional curriculum normally treats the interior angle sums of triangles and quadrilaterals first, before dealing with their exterior angle sums (and then proving them in terms of their interior angle sums). However, from a logical point of view, we could just as easily first deal with the exterior angle sums of polygons, then use them to prove the interior angle sums of polygons.

Further Exploration

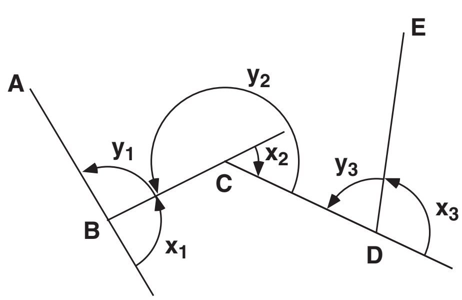
You may also want to further challenge your stronger students to try to find a general formula for the interior angle sum of *any* polygon (including crossed ones). For this purpose, you might give them figures to explore like those shown here.







GENERAL FORMULA



Imagine a turtle walking counterclockwise from A to B in the diagram shown here, turning through angle x_1, moving along segment BC, turning through angle x_2, and so on, until the figure closes and the turtle is once again facing in the direction of $\overline{AB}$. The sum of the measures of the turning angles must therefore be a multiple of 360°. Therefore,

$$\sum x_i = k \cdot 360° \ldots k = 0; 1; 2; 3; \text{etc.}$$

The sum of the measures of the interior angles is now simply the difference between $n \cdot 180°$ and the sum of the measures of the turning angles, where n is the number of vertices. Therefore,

$$S = \sum y_i = n \cdot 180° - \sum x_i = n \cdot 180° - k \cdot 360°$$
$$= 180° (n - 2k)$$

For a simple closed polygon, such as a triangle, a convex or concave quadrilateral, and so on, the total turning is $k = 1$ because we undergo one full rotation walking around its perimeter. Students can use a pen or pencil to rotate one side onto the other, continuing around the perimeter, and note the *total turning* of the pen or pencil until they return to their starting point. For example, for the star pentagon shown, $k = 2$, and for the other two figures it is respectively 2 and 3. Once the k value of a polygon has been determined, the interior angle sum can be found by substitution in the formula given above.