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Slaying a Geometrical Monster:
Finding the Area of a Crossed Quadrilateral

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Introduction

Despite new technologies such as calculators making it obsolete, for example, knowing how to calculate the square root of any number by hand, the development of new technologies often require new skills and may make certain new topics more relevant. A case in point is the ease by which learners could 'accidentally' drag a quadrilateral in dynamic geometry into the shape of a 'crossed' quadrilateral (or directed to do so), and perhaps observing that not only is Varignon's theorem that the midpoints of a quadrilateral form a parallelogram still valid, but that the area of the Varignon parallelogram *EFGH* remains half that of the original quadrilateral *ABCD* (see Figure 1).

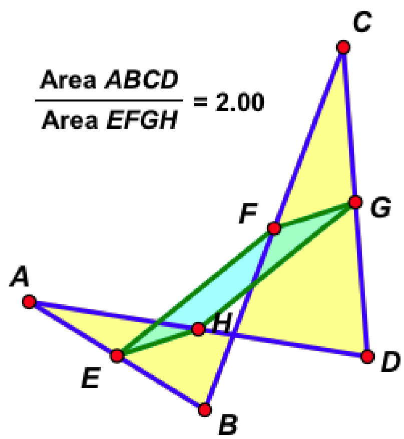

Figure 1: Varignon's theorem for a crossed quadrilateral

Though the vast majority teachers (and textbooks) are likely choose the easy way out by just ignoring a crossed quadrilateral by the Lakatosian process of 'monster-barring' by just conveniently declaring it as a 'non-quadrilateral', a very interesting and informative opportunity for further investigation is unfortunately lost.

Clearly there are questions seriously begging for an explanation: How is the area of a crossed quadrilateral determined by the software? Why is the ratio between the two areas still two?

In this sense, one would pedagogically be using the computer software as a kind of 'black box' to generate surprise and wonder, hopefully to stimulate students' curiosity further, and creating *intellectual needs* for both *causality* (explanation) and *certainty* to use the terminology of the framework of Harel (2013) and others.

Proving that *EFGH* is a parallelogram in the case when *ABCD* is crossed is left to the reader, since it is quite easy to prove in the same way as for convex and concave quadrilaterals (except that both diagonals *AC* and *BD* fall outside). We shall now proceed to first prove the area relationship for the convex and concave cases before proceeding to consider determining the area of a crossed quadrilateral.

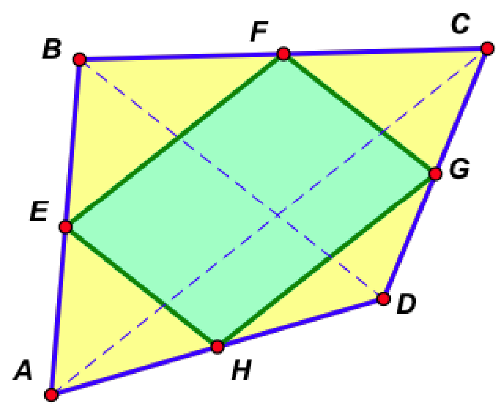

Figure 2: Convex case

Proving the area relationship

There are several different ways of proving this result for the convex case, of which we will only consider one approach, since it generalizes more easily. What follows below is based on ideas from Coxeter & Greitzer (1967), and is essentially taken from the Teacher Notes in De Villiers (1999a). It might be instructive for one's students to first work through hints like those given below with reference to Figure 2, which might help guide them to *explaining why* (prove that) the result is true for the convex case.

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Hints

1. Express the area of *EFGH* in terms of the area of *ABCD* and the areas of triangles *AEH*, *CFG*, *BEF* and *DHG*.
2. Drop a perpendicular from *A* to *BD*, and express the area of triangle *AEH* in terms of the area of triangle *ABD*.
3. Similarly, express the areas of triangles *CFG*, *BEF* and *DHG* respectively in terms of the areas of *CBD*, *BAC* and *DAC*, and substitute in 1.
4. Simplify the equation in 3 above to obtain the desired result.

Proof

1. Using the notation (*XYZ*) for the area of polygon *XYZ* we have: (*EFGH*) = (*ABCD*) - (*AEH*) - (*CFG*) - (*BEF*) - (*DHG*).
2. If the height of Δ*ABD* is *h*, then (*ABD*) = ½**BD***h* and (*AEH*) = ½*(½**BD*)* ½**h* = ¼*(*ABD*), or simply base and height are half those of large triangle.
3. (*EFGH*) = (*ABCD*) - ¼(*ABD*) - ¼(*CBD*) - ¼(*BAC*) - ¼(*DAC*).
4. (*EFGH*) = (*ABCD*) - ¼(*ABCD*) - ¼(*ABCD*) = ½ (*ABCD*).

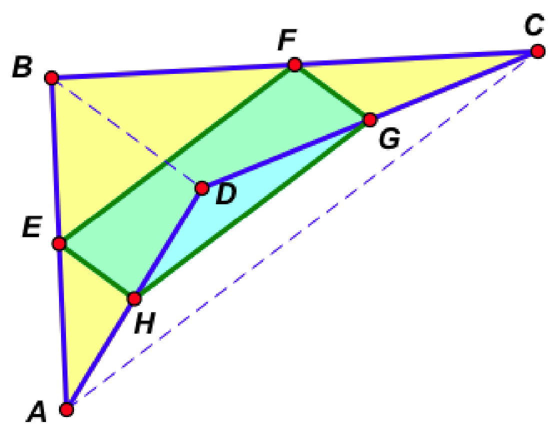

Figure 3: Concave case

Further Discussion

You may also want your students to work through an explanation for the concave case, as it is generically different. For example, unless the notation is carefully reformulated (e.g. see crossed quadrilaterals below), the equation in 1 does not hold in the concave case, but becomes (*EFGH*) = (*ABCD*) - (*AEH*) - (*CFG*) - (*BEF*) + (*DHG*) (see Figure 3). However, substituting into this equation as before, and simplifying, leads to the same conclusion.

Area of Crossed Quadrilaterals

Varignon's theorem is also true for crossed quadrilaterals *ABCD* that *EFGH* is half its area, as mentioned and illustrated in Figure 1 above. This obviously requires careful consideration of what we mean by the area of a crossed quadrilateral. Let us now first carefully try and define a general area formula for convex and concave quadrilaterals, and then *consistently* apply it to a crossed quadrilateral.

It seems natural to define the area of a convex quadrilateral to be the sum of the areas of the two triangles into which it is decomposed by a diagonal. For example, diagonal *AC* decomposes the area as follows (see 1st figure in Figure 4): (*ABCD*) = (*ABC*) + (*CDA*).

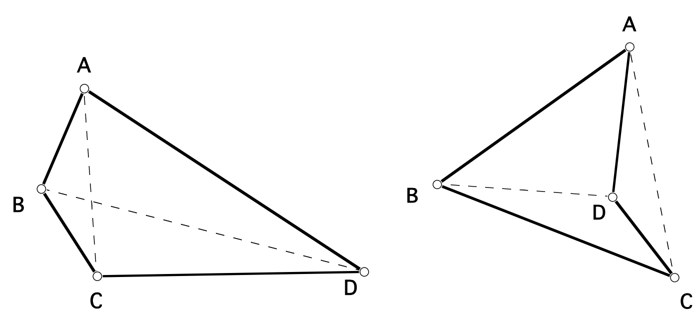

Figure 4: Decomposing a quadrilateral into triangles

In order to make this formula also work for the concave case (see 2nd figure in Figure 4) we obviously need to define (*CDA*) = - (*ADC*). In other words, we can regard the area of a triangle as being *positive* or *negative* according as its vertices are named in *counterclockwise* or *clockwise* order (or vice versa). For example:

(*ABC*) = (*BCA*) = (*CAB*) = -(*CBA*) = -(*BAC*) = -(*ACB*).

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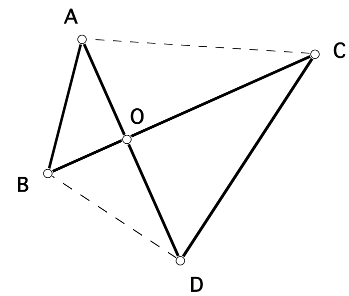

Figure 5: Decomposing a crossed quadrilateral

Applying the above formula and definition of area in a crossed quadrilateral (see Figure 5) we find that diagonal *AC* decomposes its area as follows:

(*ABCD*) = (*ABC*) + (*CDA*) = (*ABC*) - (*ADC*).

In other words, this formula forces us to regard the *"area"* of a crossed quadrilateral as (the absolute value of) the *difference* between the areas of the two small triangles *ABO* and *ODC*. {Note that diagonal *BD* similarly decomposes (*ABCD*) into (*BCD*) + (*DAB*) = -(*DCB*) + (*DAB*)}. An interesting consequence of this is that a crossed quadrilateral will have zero *"area"* if the areas of triangles *ABO* and *ODC* are equal, as dragging with dynamic geometry will easily confirm.

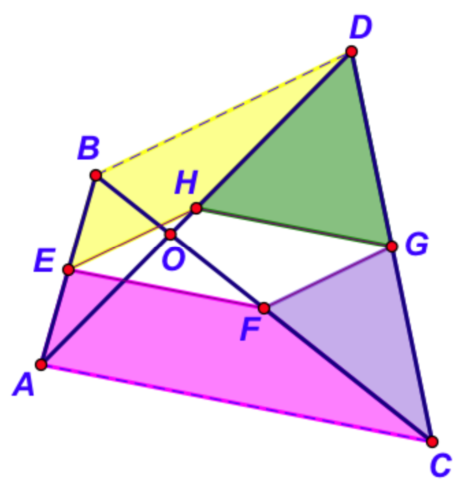

Figure 6: Proof of Varignon's area relationship

We can now determine the area of the Varignon parallelogram *EFGH* of the crossed quadrilateral *ABCD* in Figure 6 as follows (and assuming here for convenience that the area of the clockwise labeled *ABCD* is positively signed):

$(EFGH) = (ABDC) – (EBDH) – (AEFC) – (CFG) – (HDG)$

$= (ABDC) – ¾ (ABD) – ¾ (ABC) – ¼ (CBD) – ¼ (ADC)$

$= (ABDC) – ½(ABD) – ½(ABC) – ¼[(ABC) + (CBD)] – ¼[(ABD) + (ADC)]$

$= (ABDC) – ½ (ABD) – ½ (ABC) – ½ (ABDC)$

$= ½[[(ABDC) – (ABD)] – (ABC)]$

$= ½[(ADC) – (ABC)]$

$= ½[(ODC) – (OBA)]$

$= ½(ABCD)$ … (as shown earlier $(ODC) – (OBA)] = (ABCD)$)

However, we can also formally use the idea of *signed areas* developed earlier as follows to determine the area of *EFGH* in terms of the crossed quadrilateral *ABCD* in exactly the same way as for the clockwise-labeled convex or concave quadrilaterals *ABCD* shown with reference to Figures 2 and 3. (Note that clockwise labeling is assumed positive below, and that the second and third steps of the proof are not necessary, but merely illustrative of using the oppositely signed areas in relation to the particular example shown in Figure 6). For example,

$(EFGH) = (ABCD) - (AEH) - (FCG) - (EBF) - (DHG)$

$= (ABCD) - (AEH) + (GCF) - (EBF) + (GHD)$ … (triangles *FCG* & *DHG* become labeled counter-clockwise when clockwise, convex *ABCD* becomes crossed)

$= (ABCD) - ¼ (ABD) + ¼ (BDC) - ¼ (BCA) + ¼ (CAD)$

$= (ABCD) – ¼ (ABCD) – ¼ (ABCD)$

$= ½ (ABCD)$

The above method can also be extended to determine the areas of crossed polygons of higher order as was done to define and determine the area of the formed crossed hexagon in the investigation discussed in De Villiers (1999b).

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Software method

The reader may wonder if this is actually how the area of polygons, including crossed ones, is generally determined by dynamic geometry software. While I don't have access to the machine code, my guess is that the area is actually quite easily defined and determined by using the trapezium rule as follows (compare Nishiyama, 2013).

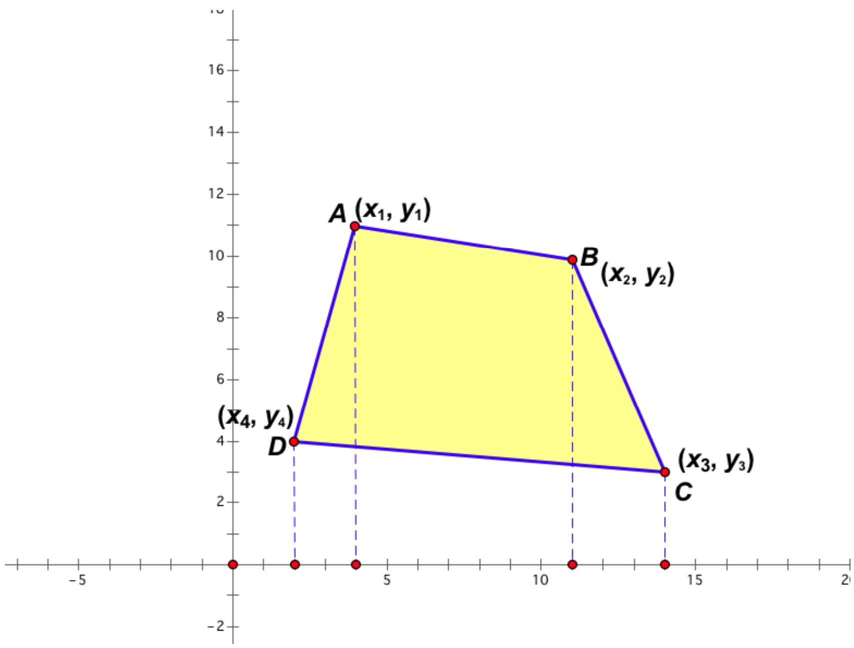

Figure 7: Using coordinates and trapezium rule

Consider Figure 7 showing a quadrilateral with the coordinates of the vertices labeled in order as (x_1, y_1), (x_2, y_2), (x_3, y_3) and (x_4, y_4). In the given diagram the area *ABCD* is clearly determined by the area under the graph *DABC* minus the area under graph *CD*. Hence, in coordinates and using the trapezium rule, the area of $ABCD = ½ [(y_1 + y_2)(x_2 – x_1) + (y_2 + y_3)(x_3 – x_2) - (y_3 + y_4)(x_3 – x_4) + (y_4 + y_1)(x_1 – x_4) = ½ [(y_1 + y_2)(x_2 – x_1) + (y_2 + y_3)(x_3 – x_2) + (y_3 + y_4)(x_4 – x_3) + (y_4 + y_1)(x_1 – x_4)$.

This cyclic formula not only generalizes to any closed polygon, and as one would expect (for consistency), it also produces the same area for a crossed quadrilateral as the earlier formula |(*ABC*) - (*ADC*)|, as experimentally illustrated in Figure 8 using *Sketchpad* by the two different measurements and calculations.

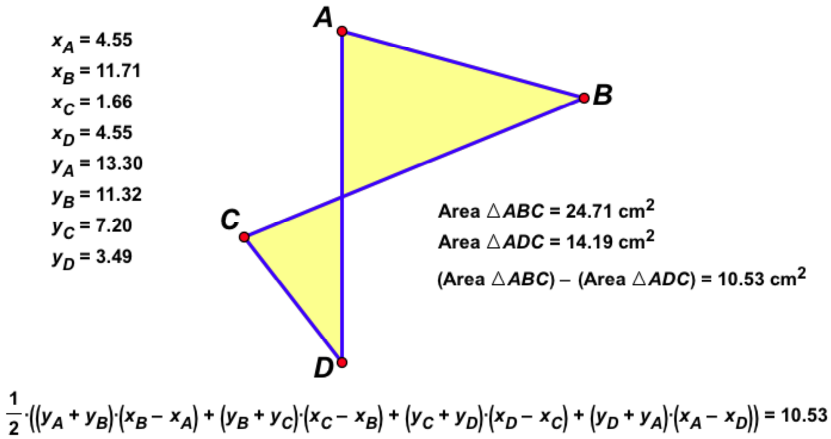

Figure 8: Experimental illustration of equivalent area formulae

Concluding remarks

This example firstly illustrates the value of looking at simpler cases of a problem. Secondly, it nicely illustrates the important general mathematical principle of maintaining *structural consistency of definitions* is when moving from a *familiar* to an *unfamiliar* context in mathematics. The same sense of maintaining structure and consistency can guide us to defining 'interior angles' for crossed polygons in general as well as the deduction of a general formula for the interior angles of *any* polygon. A surprising, counter-intuitive conclusion of such an extension is that two of the 'interior' angles of a crossed quadrilateral become *reflexive*, and that its interior angle sum is 720 degrees (De Villiers, 1999c).

The same applies when moving from operations for positive whole numbers to negative whole numbers, then further towards fractions, and eventually to complex numbers and quaternions, etc. Engaging at least one's mathematically talented students with this intriguing problem, which can quite naturally arise in dynamic geometry, could therefore provide them with the fruitful seed of a worthwhile instructive learning experience that can carry them further into their mathematical studies.

Note: A dynamic geometry sketch using *GeoGebra* to investigate and illustrate the Varignon area for crossed quadrilaterals result online, as well as other interesting properties of crossed quadrilaterals is available at: http://dynamicmathematicslearning.com/crossedquad_anglesum.html

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De Villiers, M. (1999b). A Sketchpad discovery involving areas of inscribed polygons. *Mathematics in School*, 28(1), March, pp. 18-21. (An online dynamic geometry sketch illustrating these results is available at: http://dynamicmathematicslearning.com/area-inscribed-polygons.html)

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