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Note:This paper is a reproduction of an earlier paper of mine with the same title in Imstusnews, 22, 3-5, 1992.

The Affine Equivalence of Cubic Polynomials

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What follows is a short heuristic description of the way in which I made the personal, interesting discovery some ten years ago that all cubic polynomials are affine equivalent, a result which should be easily accessible and interesting to most high school learners. This result should be given some consideration in our new FET Mathematics curriculum, which now contains sufficient transformation geometry for it to be easily accessible. Moreover, it provides an opportunity to illustrate the integration of geometric ideas with algebraic ones. Such an investigation could easily be done with *Sketchpad*, which is particularly suited for investigating transformations of graphs as it directly allows one to define and plot a new function $g(x)$ in terms of $f(x)$. So for example, $g(x) = -f(x)$ would represent a reflection in the *x*-axis.

My own personal investigation was, however, not originally done with graphing software, though use of such visualisation aids in a teaching-learning context is likely to benefit learners greatly. I started out with the following two observations that:

(1) All straight lines $y = mx + c$ are congruent (that is any two straight lines can be mapped exactly onto each other by the isometric transformations, that is reflections, rotations and translations, for example, see Figure 1).

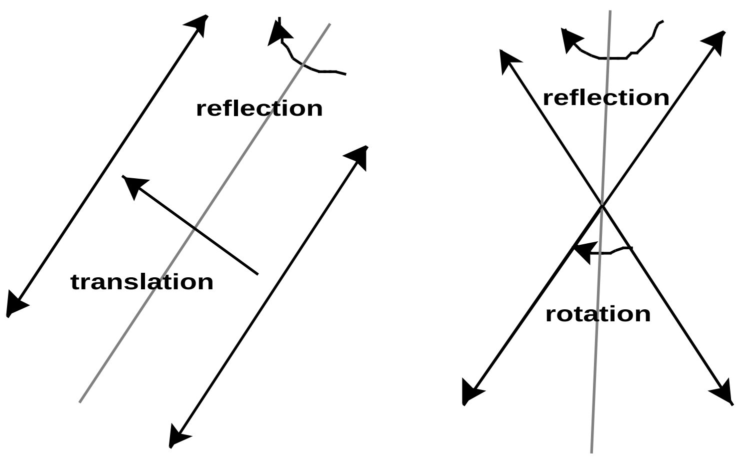

Figure 1

(2) All parabola $y = ax^2 + bx + c$ are similar (that is, any two parabola can be mapped exactly onto each other by a combination of the similar transformations, i.e. enlargements or reductions in combination with the isometries, for example see Figure 2). For a proof of this result, consult De Villiers (1994).

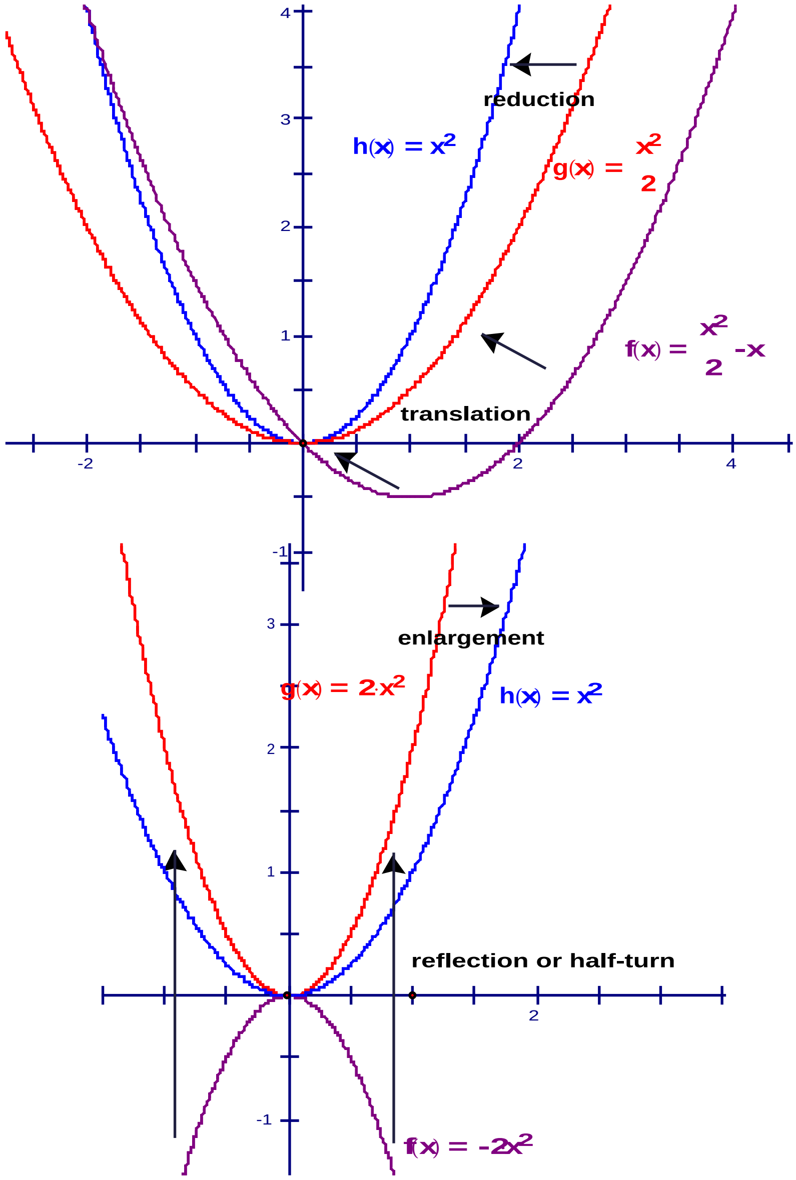

Figure 2

With Felix Klein's famous *Erlangen*–approach in mind regarding a hierarchical classification of geometry into the study of the following groups of transformations and those properties that remain invariant under a specific group: isometries (preserves congruence), similarities (preserves similarity), affinities (preserves parallelness of lines), the following conjecture then seemed obvious to make:

> All cubic polynomials $y = ax^3 + bx^2 + cx + d$ are affine equivalent (that is, any two cubic polynomials can be mapped exactly onto each other by a combination of the affine transformations, that is shears, stretches or the general linear transformations in combination with the similarities).

In order to analyse the conjectured affine equivalence of cubic polynomials, I first thought of simplifying the problem by just considering two special affine transformations namely stretching and shearing to the x - or y -directions.

The transformation equations for a stretch and a shear in the x – and y – directions can easily be derived by learners by considering the effect of these transformations as shown in Figure 3. Figures 3(a) and 3(b) respectively show the effect of a stretch and a shear on a square in the x -direction.

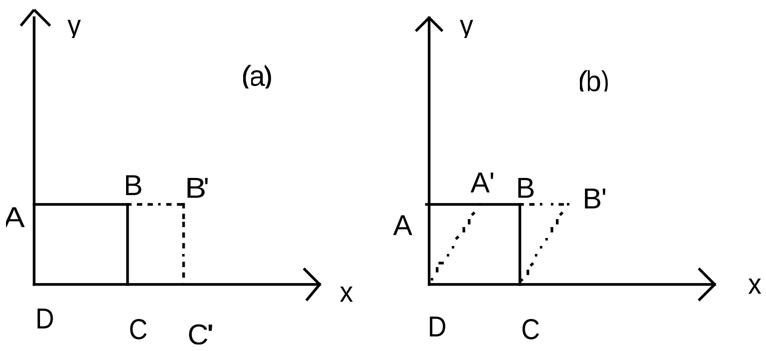

FIGURE 3

The transformation equations are given by:

***x* -direction**		***y* -direction**	
Stretch:	$x' = kx$	Stretch:	$x' = x$
	$y' = y$		$y' = ky$
Shear:	$x' = x + ky$	Shear:	$x' = x$
	$y' = y$		$y' = y + kx$

Personally I find it useful when working with the transformation of functions to substitute the solutions of the transformations for x and y in terms of x' and y' into the general formula $y = f(x)$ to find the corresponding transformed equation $y' = f(x' \pm y')$. In transforming an equation $y = f(x)$ to another $y = g(x)$ by the above mentioned transformations, one should therefore remind oneself that the latter equation is actually $y' = g(x')$.

Using this substitution, the stretch and shear transformations in the x - and y - directions can be written as shown below (dropping the primes), and is left as an exercise for the reader:

	***x*-direction**		***y*-direction**
Stretch:	$y = f\left(\frac{x}{k}\right)$	Stretch:	$y = kf(x)$
Shear:	$y = f(x - ky)$	Shear:	$y = f(x) + kx$

As cubic polynomials can be classified into 3 distinctly dissimilar types of curves as shown in Figure 4, I was faced with the problem of finding an affine transformation which would transform the one type into the other. Stretches could be dismissed as they would not affect the basic shape of each type, for example, stretching $y = x^3$ in the x - or y -directions respectively gives $y = kx^3$ or $y = \frac{x^3}{k^3}$, thus retaining the horizontal point of inflection. The same applies to the other two types as can easily be checked by the reader.

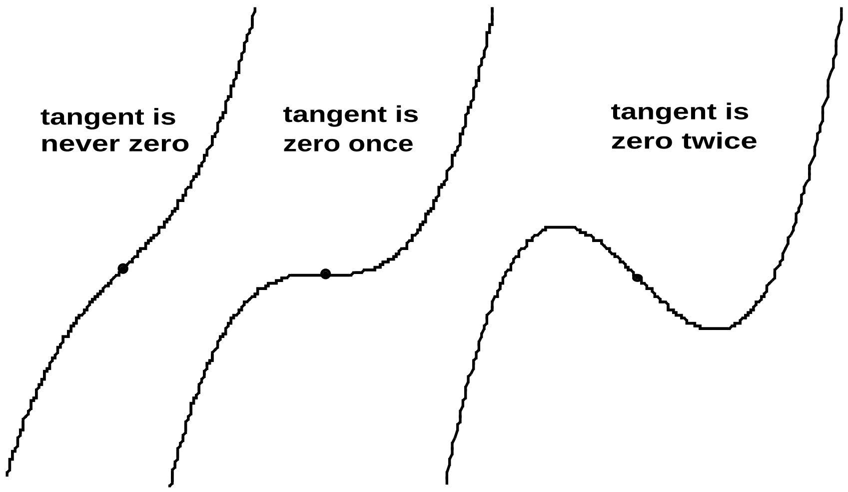

Figure 4

Shearing in the x -direction on the other hand introduces higher order terms of y so that we no longer have a polynomial. This only left the possibility of shearing in the y -direction (or some other more general affine transformation). But how was it possible that the simple shear $y = f(x) + kx$ could transform the one type into the other? Although initially sceptical, I nevertheless proceeded investigating it with a particular example, but then suddenly recalled some years ago proving that all cubic polynomials are point symmetric (see De Villiers, in press). In this article it is shown how the general cubic polynomial $y = ax^3 + bx^2 + cx + d$ is transformed to $y = ax^3 + \left(c - \frac{b^2}{3a}\right)x$ by translating it so that its point of inflection falls at the origin, for example $\frac{b}{3a}$ units in the x -direction and $-f\left(\frac{-b}{3a}\right)$ units in the y -direction[1]. Then this transformed equation (which can be either one of the three types in Figure 4 above) is shown to be point symmetric with respect to the origin.

Next I noticed that by choosing $k = -\left(c - \frac{b^2}{3a}\right)$ in $y = f(x) + kx$ for a shear in the y -direction this transformed equation $y = ax^3 + \left(c - \frac{b^2}{3a}\right)x$ simply becomes $y = ax^3$ (and the reader is again invited to work through this substitution). The latter can in turn can be stretched with the value of $k = 1/a$ in the y -direction to map exactly onto $y = x^3$. This then completed the proof of the affine equivalence of cubic polynomials!

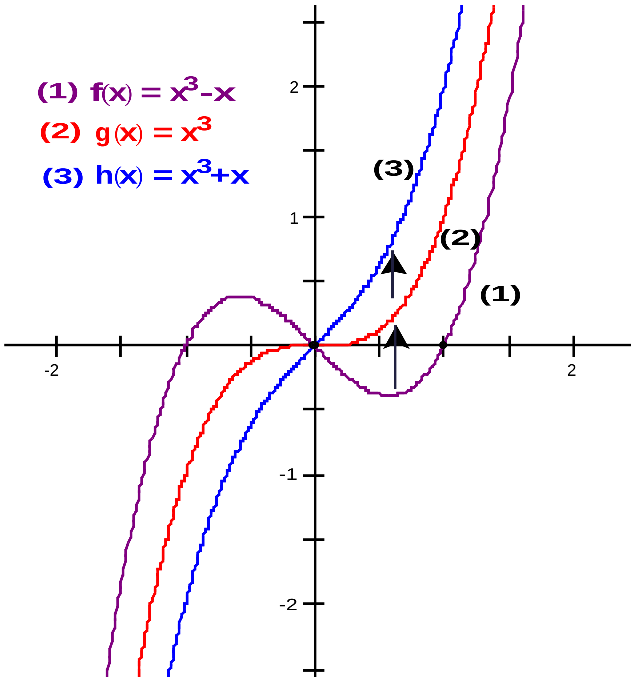

Figure 5

Figure 5 for example illustrates how $y = x^3 - x$ and $y = x^3$ are respectively sheared in the y -direction, into $y = x^3$ and $y = x^3 + x$, by the addition of x -terms on the right hand sides of the equations. A zipped *Sketchpad* sketch that can be used as a dynamic

[1] A general translation of p units in the x -direction and q units in the y -direction is given by $y = f(x - p) + q$. The reader is invited to work through this substitution as an exercise.

demonstration by a teacher or exploration by students of the translation, shearing and stretching of a general cubic polynomial to map onto $y = x^3$ can be downloaded from:

http://mzone.mweb.co.za/residents/profmd/cubic.zip

References

De Villiers, M. (In press). All cubic polynomials are point symmetric. *Learning and Teaching Mathematics.*

(A copy of this article can be directly downloaded from http://mzone.mweb.co.za/residents/profmd/cubics.pdf)

De Villiers, M. (1994). All parabola similar? Never! *Pythagoras*, 34, 26-30.

(A copy of this article can be directly downloaded from http://mzone.mweb.co.za/residents/profmd/simpara.pdf)

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"*Algebra is the offer made by the devil to the mathematician. The devil says: 'I will give you this powerful machine, and it will answer any question you like. All you need to do, is give me your soul: give up geometry ...* '"

- Sir Michael Atiyah (2001)

"*... mathematics is like a diamond: extremely hard material, but valuable and highly prized both for its industrial applications and for its intrinsic beauty*."

- Harold P. Boas (Ed., *AMS Notices*, 2003)

"*Mathematical arguments, on the other hand, are self-contained and ineluctable. They are not contextualized by anything outside themselves so that, in particular, once the premises are clarified and accepted, the conclusions become inevitable ...*

There is economy in a good mathematical argument, and different ways of approaching an investigation may cast more light but can never contradict. Mathematicians try to identify and strip away any hypothesis not essential to a conclusion; this leads to results of astonishing generality that are hard to grasp by a neophyte who may become distracted by irrelevant details."

- Ed Barbeau in a Book Review in *AMS Notices*, September 2002, p. 906.