Published in *Learning & Teaching Mathematics*, No. 1, April 2004, pp. 12-15. *LTM* is a publication of AMESA at http://www.amesa.org 

# All cubic polynomials are point symmetric

Note:This paper is a reproduction of an earlier paper of mine with the same title in Imstusnews, 19, 15-16, Nov 1989.

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Traditional textbooks usually discuss the point symmetries of functions only in relation to the origin. For instance, a graph is considered point symmetrical in relation to the origin *O* when each point *P* of a graph as shown in Figure 1, has a corresponding point *Q* (also on the graph) under a reflection through *O* so that $PO = OQ$. Or equivalently in the plane, each point *P* of the graph can be mapped onto a corresponding point *Q* (also on the graph) by means of a half-turn (a rotation through 180 degrees) around *O*.[1] Using the transformation formula for a half turn, it therefore follows that a graph is point symmetric in relation to the origin if $y = f(x) \Leftrightarrow y = -f(-x)$; in other words if it remains *invariant* under a half-turn around the origin.

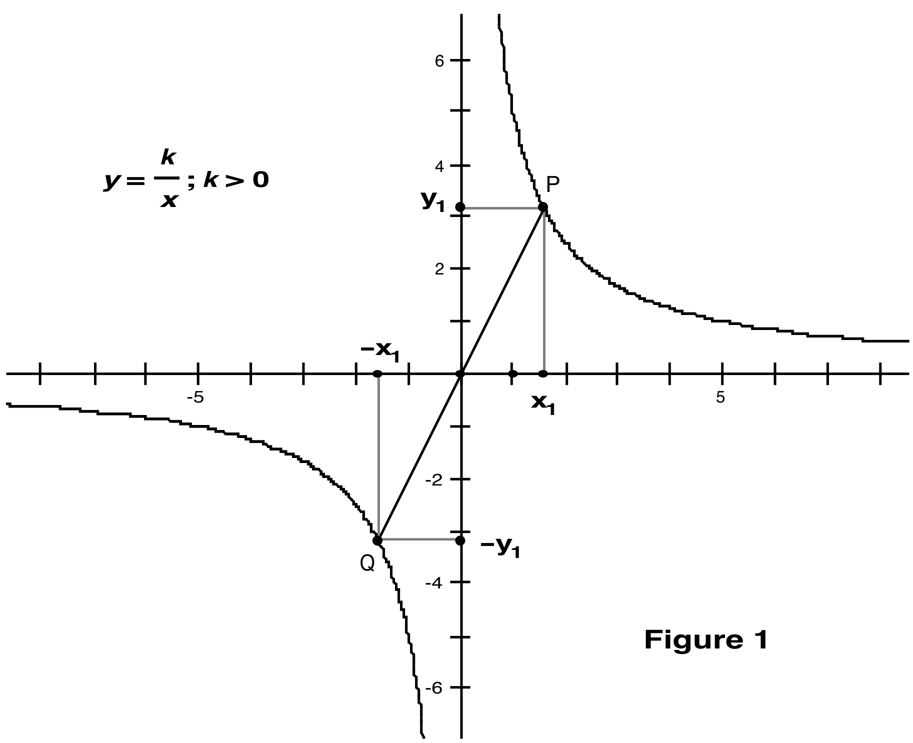


Figure 1

 

Given the recommended treatment of transformation geometry in the current proposals for the new FET curriculum in South Africa, an investigation of the point symmetries of graphs can be worthwhile. In our present school syllabi, apart from the hyperbola above, other examples of functions with point symmetries at the origin are: $y = mx$; $y = \sin x$; $y = \tan x$; $y = x^2$; $y = x^3 \pm x$.

But what of a cubic function with a point of symmetry not at the origin? For example, if one translated $y = x^3$ by one unit upwards it becomes $y = x^3 + 1$, which is clearly point symmetric around (0; 1). Similarly if one considered the translation of $y = x^3$ by one unit to the left and two units up, it becomes $y = (x - 1)^3 + 2$. This graph is clearly point symmetrical around (1; 2), since we can simply translate it back to the origin to obtain $y = x^3$, which is point symmetric.

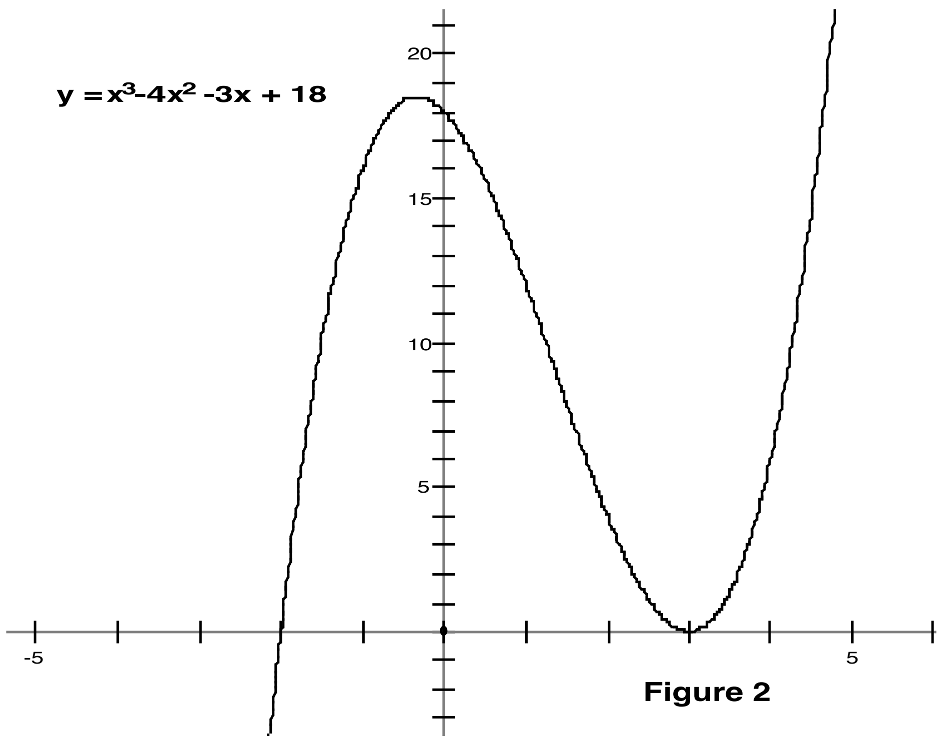


**Figure 2**

Now consider the following cubic polynomial, namely $y = x^3 - 4x^2 - 3x + 18 = (x + 2)(x - 3)^2$, the graph of which is shown in Figure 2. Intuitively one may visualise a point of symmetry approximately between $x = 1$ and $x = 2$. But how could one find this point of symmetry exactly if it existed, and prove that it was one?

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Let us firstly look at the gradient of the function. If we move from left to right on the curve, we can see that the gradient starts from a large positive value, decreases to zero, becomes negative and reaches a maximum negative value before starting to increase again to become zero and large positive again. By roughly drawing the derivative $\frac{dy}{dx}$ = $3x^2$ - $8x$ - 3 on the same axis, it seems obvious that the point of symmetry must be on the axis of symmetry of the derivative, i.e. at the point of inflection of the cubic. The substitution of this value of the derivative, namely, $x = \frac{4}{3}$ into the original function gives us the hypothesised point of symmetry, namely, $(\frac{4}{3}, \frac{250}{27})$. To prove that this is indeed a point of symmetry we now only need to translate the function to the origin and to test for point symmetry around the origin. For example, the transformed equation using the standard formula for a translation is:

$$y = (x + \frac{4}{3} + 2)(x + \frac{4}{3} - 3)^2 - \frac{250}{27}$$

Which simplifies to:

$$y = x^3 - \frac{25}{3}x$$

However, this equation is clearly equivalent to

$$y = -((-x)^3 - \frac{25}{3}(-x))$$

and therefore completes the proof that the point $(\frac{4}{3}, \frac{250}{27})$ is a point of symmetry of the original function.

To make sense of this proof, some learners may perhaps have to be reminded that a translation is an isometric transformation, which means that it preserves congruency, and that the graph of the transformed function is therefore geometrically congruent to the original graph, and all symmetrical properties are therefore preserved.

It is now left to the reader to consider the function $y = x^3 - x^2$ and show that it has a point of symmetry at its inflection point $(\frac{1}{3}; \frac{-2}{27})$. After learners have investigated a few of

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more cubic functions, it is likely that some may conjecture that all cubic polynomials are point symmetric. With some guidance, learners ought to be able to come up with a general proof more or less as follows.

**Proof**

Consider a general cubic polynomial $y = ax^3 + bx^2 + cx + d$

$$\text{Thus, } \frac{dy}{dx} = 3ax^2 + 2bx + c$$

The *x* co-ordinate of the inflection point lies on the axis of symmetry the derivative and is therefore given by $x_{sim} = \frac{-b}{3a}$. Through substitution we can also easily determine the corresponding $y$ – value, for example:

$$\begin{aligned} y_{sim} &= -\frac{b^3}{27a^2} + \frac{b^3}{9a^2} - \frac{bc}{3a} + d \\ &= \frac{2b^3}{27a^2} - \frac{bc}{3a} + d \end{aligned}$$

Consider now the translation of the function $y = f(x + x_{sim}) - y_{sim}$ so that the inflection point coincides with the origin:

$$y = a\left(x - \frac{b}{3a}\right)^3 + b\left(x - \frac{b}{3a}\right)^2 + c\left(x - \frac{b}{3a}\right) + d - y_{sim}$$

After simplification this gives :

$$y = ax^3 + \left(c - \frac{b^2}{3a}\right)x \qquad \ldots (1)$$

which is clearly point symmetric around (0; 0), since

$$y = -\left[a(-x)^3 + \left(c - \frac{b^2}{3a}\right)(-x)\right]$$

remains equivalent to the transformed equation (1). This then completes the proof of the conjecture.

In my view, an exploration of the point symmetry of cubic polynomials using graphing software such as *Sketchpad* provides a very nice setting for an active investigation by secondary school learners. A zipped Sketchpad sketch that might be useful can be downloaded directly from http://mzone.mweb.co.za/residents/profmd/cubic.zip

 

Such an investigation could provide learners with the opportunity to formulate a conjecture on their own and create a need for them to attempt to prove it themselves. Furthermore, besides bringing some nice geometric ideas to algebra, it also shows a somewhat different application of differentiation, and that not only the roots, but the turning points of the derivative, may give us useful information.

As a possible project for a mathematical investigation, more ambitious learners at the Additional Mathematics level may also be encouraged to investigate the general relationship between symmetric functions and their derivatives (e.g. see De Villiers, 1991). Or alternatively perhaps to derive formulae for the axes and points of symmetry of the general conics $px^2 + 2qxy + ry^2 + 2sx + 2ty + u = 0$ (e.g. see De Villiers, 1993). Another interesting property of cubic polynomials is that of "*affine equivalence*" and a companion article about this can be downloaded directly from:

http://mysite.mweb.co.za/residents/profmd/cubeaffine.pdf

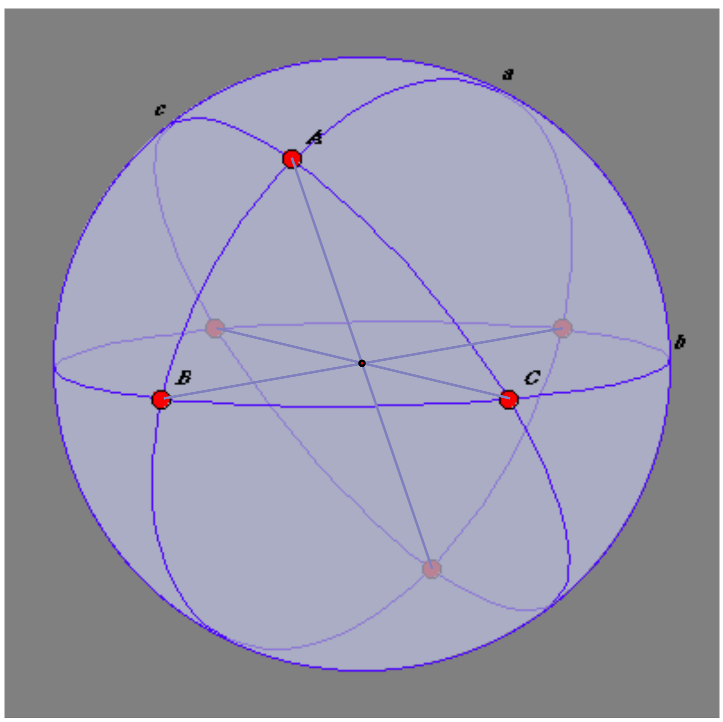


**Figure 3**

**Note**

1. Though the different concepts of point symmetry and half-turn symmetry are mathematically equivalent in the plane, it is very important to note that this is not necessarily the case in space. For example, some three dimensional solids (like some crystals) have point symmetry, but not half-turn symmetry. Perhaps the simplest example is found on the surface of a sphere. Visualise a triangle on the surface of a sphere and its antipode, with lines joining the corresponding vertices (see Figure 3). Clearly the triangle and its antipode are point symmetric with respect to the centre of the sphere, but a half-turn in the plane through $B$ and $C$ and their antipodes will map $B$ to $B'$ and $C$ to $C'$, but not $A$ to $A'$.

**References**

De Villiers, M. (1991). Vertical line and point symmetries of differentiable functions. *International Journal for Mathematical Education in Science and Technology*, 22(4), 621-644.

De Villiers, M. (1993). The affine invariance and line symmetries of the conics. *Australian Senior Mathematics Journal*, 7(2), 32-50.
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