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An interesting cyclic quadrilateral result

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A few months ago I experimentally discovered the following interesting result as shown in Figure 1 using the dynamic geometry software *Sketchpad*:

"The respective intersections *E*, *F*, *G* and *H* of the angle bisectors of angles *A*, *B*, *C* and *D* of a cyclic quadrilateral *ABCD,* with the circumcircle, form a rectangle."

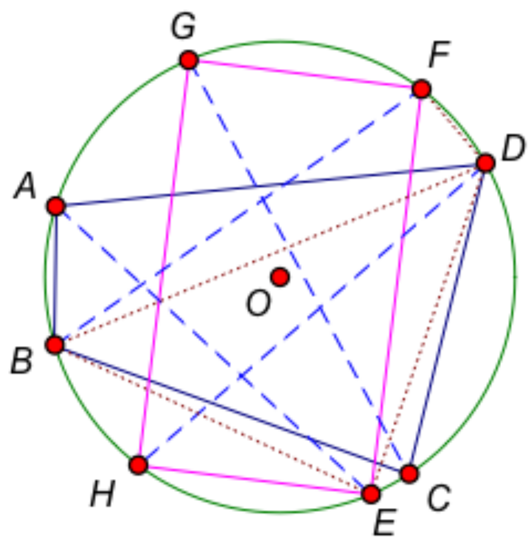

Figure 1

The result follows directly from a useful Lemma proved on p. 190 of my book *Some Adventures in Euclidean Geometry* in the process of proving another result. The Lemma basically states that the angle bisector of *BAD* intersects the circum circle at *E*, the midpoint of arc *BHCD*. Let us now prove the Lemma as well as the desired result.

Proof

In Figure 1, connect *E* with *B* and *D*, and *B* with *D*. Since angle *BAE* = angle *DAE*, it follows that chord *BE* = chord *DE* (equal angles subtend equal chords). Hence, *E* is the midpoint of arc *BHCD*. Similarly, *G* is the midpoint of arc *BAFD*. Hence, *E* and *G* are diametrically opposite each other and lie on the diameter through *O*. Similarly, points *F*

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and H lie on the same diameter through O. Thus, since its diagonals are equal and bisect each other, $EFGH$ is a rectangle.

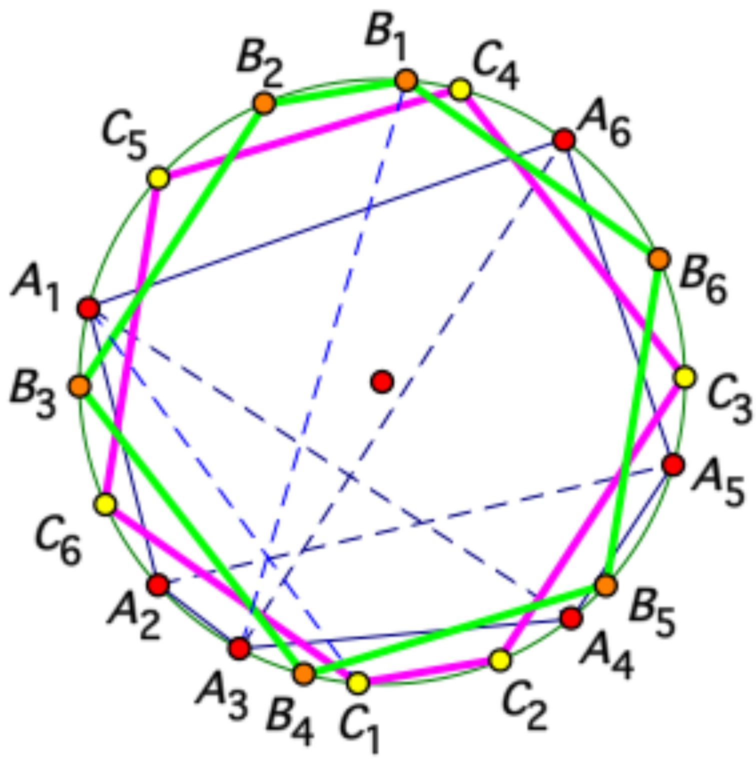

Figure 2

Further generalization

The result can be further generalized and applied to a cyclic hexagon by subdividing a cyclic hexagon $A_1A_2A_3A_4A_5A_6$ into six cyclic quadrilaterals $A_1A_2A_3A_4$, $A_2A_3A_4A_5$, $A_3A_4A_5A_6$, $A_4A_5A_6A_1$, $A_5A_6A_1A_2$ and $A_6A_1A_2A_3$. In quadrilateral $A_1A_2A_3A_4$, construct the respective angle bisectors of $\angle A_2A_1A_4$ and $\angle A_4A_3A_2$ to intersect the circum circle respectively at C_1 and B_1 as shown in Figure 2.

Then from the aforementioned result, C_1 and B_1 are diametrically opposite points lying on the diameter through the centre. Therefore, repeating the same constructions for the other five cyclic quadrilaterals in a cyclic fashion, we obtain two hexagons $B_1B_2B_3B_4B_5B_6$ and $C_1C_2C_3C_4C_5C_6$ with corresponding vertices diametrically opposite each other. In other words, the one hexagon can be given a half-turn (a rotation by 180º) to map onto the other.

Hence, if we connect any set of 12 corresponding points from these two hexagons in a cyclic fashion, we obtain a 12-gon with opposite sides equal and parallel, since the configuration has half-turn symmetry. For example, $B_1B_2C_5B_3C_6B_4C_1C_2B_5C_3B_6C_4$, and other similar sets, all have half-turn symmetry, and therefore opposite sides are equal and parallel.

In a similar fashion, the result can be further generalized to cyclic octagons, cyclic decagons, etc., by subdividing them also into quadrilaterals, but that's left as an exercise to the reader.

Reference

De Villiers, M. (2009). *Some Adventures in Euclidean Geometry*. Lulu Publishers. (Available as PDF download or in printed book form at http://www.lulu.com/content/7622884).

Note

A Java applet to dynamically illustrate the results above are available at:
http://frink.machighway.com/~dynamicm/cyclic-quad-rectangle-result1.html