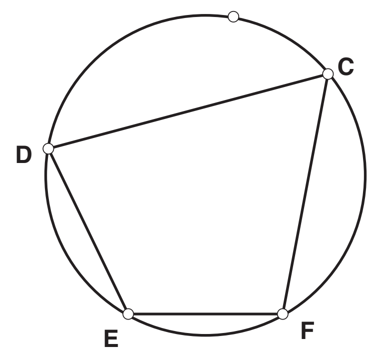
Name(s):

# Cyclic Quadrilateral

In this activity, you will investigate some properties of a quadrilateral inscribed in a circle—in other words, a quadrilateral whose vertices lie on a circle. Such a quadrilateral is called a *cyclic quadrilateral.*

## CONJECTURE




- Open the sketch **Cyclic Quad.gsp.**
- Use Sketchpad's calculator to sum each pair of opposite angles.

**1.** Drag a vertex. What can you say about the two pairs of opposite angles of a cyclic quadrilateral? (You may have dragged the vertex enough to cross the sides of the quadrilateral. Do not worry about these "crossed quadrilaterals" for now.)

**2.** Press the *Show perpendicular lines* button. What do you notice about the perpendicular bisectors of a cyclic quadrilateral?

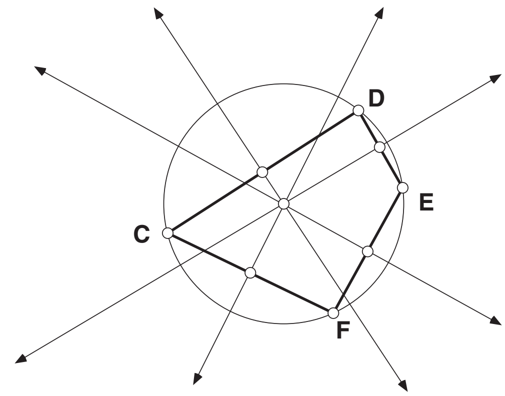
### Further Exploration

Can you drag your cyclic quadrilateral into the general shapes of some of the special quadrilaterals that you've already seen? For example, try making a kite, an isosceles trapezoid, a parallelogram, a rhombus, a rectangle, and a square.

## EXPLAINING

In the preceding section, you discovered the following:

- Opposite angles of a cyclic quadrilateral are supplementary (as long as the quadrilateral is not crossed).
- The perpendicular bisectors of the sides of a cyclic quadrilateral always remain concurrent at the center of the circle. This center is called the *circumcenter* of the cyclic quadrilateral.

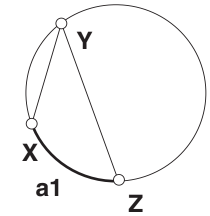



m∠CDE + m∠EFC = 180.0°
m∠DEF + m∠FCD = 180.0°

**3.** You will explain the second conjecture first. Write an explanation below for the concurrency of the perpendicular bisectors of a cyclic quadrilateral. (*Hint:* First explain why *each* perpendicular bisector goes through the center of the circle. Construct radii to help.)

**4.** Recall that an inscribed angle has half the measure of its intercepted arc (see the diagram). Use this result to explain why in a cyclic quadrilateral the opposite angles are supplementary. (*Hint:* In your original sketch, construct chord *DF*.)

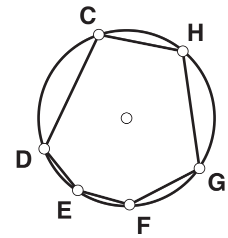
m∠XYZ = 35°




m arc angle a1 = 70°

## Further Exploration

In the cyclic hexagon shown on the right, the angles *C, E,* and *G* are called *alternate* angles. Similarly, angles *D, F,* and *H* are alternate angles.




**1.** Construct a cyclic hexagon, measure its angles, and calculate the sum of both sets of alternate angles. What do you notice?

**2.** In your own words, formulate a conjecture based on your observation in Question 1.

**3.** Can you logically explain your conjecture in Question 2? (*Hint:* Draw a diagonal so that the cyclic hexagon is divided into two cyclic quadrilaterals.)

**4.** Which are the alternate angles in a cyclic quadrilateral *ABCD*? Reformulate your earlier result for cyclic quadrilaterals in terms of alternate angles.

**5.** Can you generalize your conjecture for a cyclic hexagon further, to cyclic octagons, cyclic decagons, and so on? In other words, generalize to cyclic $2n$-gons where $n > 1$.

## CYCLIC QUADRILATERAL (PAGE 48)

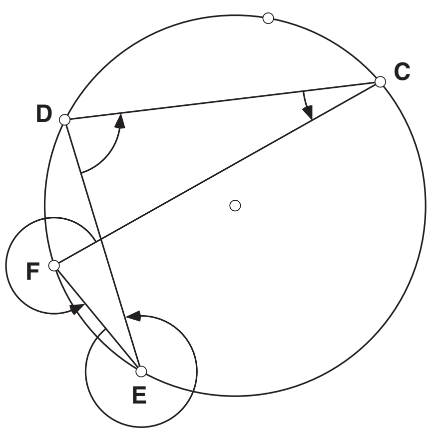
**Prerequisites:** An inscribed angle has half the measure of its intercepted arc.

**Sketch: Cyclic Quad.gsp.**

### CONJECTURE

1. The opposite angles of a (convex) cyclic quad are supplementary.
2. The perpendicular bisectors of the sides of a cyclic quad always remain concurrent.

   Some students may observe that opposite angles are no longer supplementary when the cyclic quad becomes crossed. Note that if we consider directed angles as discussed in the Crossed Quadrilateral Sum activity, the sums of the two pairs of opposite angles in a crossed cyclic quad are both equal to 360° (see figure). Therefore, in general we can say that for *any* cyclic quad, the sums of the two pairs of opposite angles are equal.



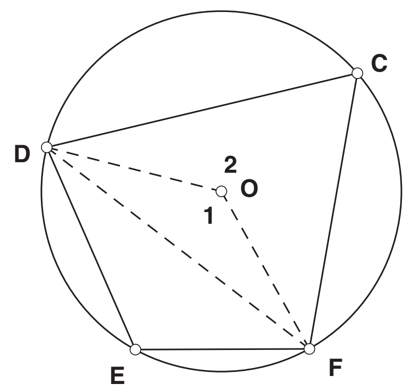

### Further Exploration

We can obtain as special cases a general isosceles trapezoid, a general rectangle, and a square, but not a general parallelogram or a general rhombus. Students might also obtain a certain general kite (when its axis of symmetry passes through the center of a circle), and you might also ask them to investigate and explain its property of having one pair of opposite right angles (angles in a semicircle).

### EXPLAINING

3. The circumcenter is equidistant from all four vertices (radii are equal), but each perpendicular bisector is the locus of all the points equidistant from the endpoints (vertices) of each side. Therefore, each perpendicular bisector must pass through the circumcenter.

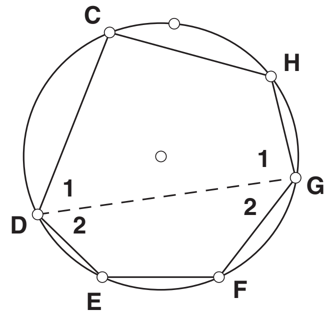
   Conversely, you may want to point out that this is a very useful condition for any polygon to be inscribed in a circle (to be cyclic). For example, for any polygon to have a circumcircle, it must have a point that is equidistant from all the vertices. Therefore, the perpendicular bisectors must meet in a single point; that is, they must be concurrent.




4. Consider the convex cyclic quadrilateral shown above. Central angle $O_1$ and inscribed angle $C$ intercept the same arc, so $m\angle O_1 = 2 \cdot m\angle C$. Likewise, $m\angle O_2 = 2 \cdot m\angle E$. But $m\angle O_1 + m\angle O_2 = 360°$; therefore $2 \cdot m\angle E + 2 \cdot m\angle C = 360°$, which reduces to $m\angle E + m\angle C = 180°$. Similarly, it can be shown that angles $D$ and $F$ are supplementary.

### Further Exploration

1. They are both equal (to 360°).
2. The two sums of the measures of the sets of alternate angles of a cyclic hexagon are equal. (This result is also true for certain types of crossed cyclic hexagons, provided that we work with directed angles, as discussed in the Crossed Quadrilateral Sum activity.)




3. Consider the figure shown above. $m\angle C + m\angle G_1 = 180°$, and $m\angle E + m\angle E + m\angle G_2 = 180°$. Therefore, $m\angle C + m\angle E + m\angle G = 360°$. Similarly, it can be shown that the sum of the measures of the other set of alternate angles is also 360°.
4. Angles *A* and *C*, and angles *B* and *D*, are the two sets of alternate angles. The two sums of the measures of the sets of alternate angles of a cyclic quadrilateral are equal.
5. In general, for certain types of cyclic 2*n*-gons where $n > 1$, the two sums of the measures of the sets of alternate angles are equal. (For convex ones, these sums are equal to $180°(n - 1)$.) The following theorem in this regard is proved in de Villiers (1996, 183–187):

   *If $A_1A_2 \ldots A_{2n}$ ($n > 1$) is any cyclic 2n-gon in which vertex $A_1 \rightarrow y\, A_{1+k}$ (vertex $A_i$ is joined to $A_{i+k}$), the two sums of alternate interior angles are each equal to $m\pi$ (where $m = n - k$ and k is the total turning we would undergo by walking around the perimeter of the polygon).*

# THE CENTER OF GRAVITY OF A TRIANGLE

(PAGE 51)

This activity introduces students to the idea that even after a result has been found to be true experimentally, creating a logical explanation for the result can be an intellectual challenge similar to the solution of a crossword or other puzzle. Experience with students has shown that they find it far more reasonable to accept that some people (for example, mathematicians) could be motivated by such intellectual challenges and activities than to accept that they would indulge in such an activity simply for the sake of verification.

You could further point out that different people have different interests. For example, not everybody is excited by bungee jumping, mountaineering, marathon running, crossword puzzles, athletics, golf, bowling, cooking, tennis, or any given activity. This does not mean that someone who is not strongly motivated in any of these areas could not have some appreciation and respect for those who have mastered, and found meaning in, a particular discipline. The challenge is therefore to inculcate and encourage some appreciation and understanding of the discipline of mathematics (and in particular of deductive reasoning) in those who do not aspire to become mathematicians or who will not seriously apply mathematics in later life.

**Prerequisites:** Midpoint triangle theorem, properties of parallelograms, triangle area formula.

**Sketches: Triangle Median.gsp** and **Centroid.gsp.** Additional sketches are **Quad Centroid.gsp, Ceva.gsp, Ceva Concurrency.gsp,** and **Ceva Pentagon.gsp.**

## CONJECTURE: LOCATING THE CENTROID

1. Segment *DE* is parallel to $\overline{AC}$.
2. The midpoint of $\overline{DE}$.
3. Responses will vary.
4. It is a straight line from the midpoint of $\overline{AC}$ to vertex *B*.
5. If the triangle is made up of many thin segments parallel to $\overline{DE}$, each of them will have its center of gravity along the path of point *F*, and therefore the