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Generalizations of some famous classical Euclidean geometry theorems

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Abstract. In this note, we introduce generalizations of some famous classical Euclidean geometry theorems. We use problems in order to state the theorems.

Keywords. Euclidean geometry, Triangle geometry.

Mathematics Subject Classification (2010). 51-04, 68T01, 68T99.

Problem 1 ([1],[2], A generalization the Lester circle theorem associated with the Neuberg cubic)**.** *Let P be a point on the Neuberg cubic. Let P_A be the reflection of P in line BC, and define P_B and P_C cyclically. It is known that the lines AP_A, BP_B, CP_C concur. Let $Q(P)$ be the point of concurrence. Then two Fermat points, P, $Q(P)$ lie on a circle.*

Remark: *Let P is the circumcenter, it is well-know that $Q(P)$ is the Nine point center, the conjucture becomes Lester circle theorem, you can see the Lester circle theorem in(*[3],[4]*).*

Problem 2 ([5],[6], A generalization of the Parry circle theorem associated with two isogonal conjugate points)**.** *Let a rectangular circumhyperbola of ABC, let L be the isogonal conjugate line of the hyperbola. The tangent line to the hyperbola at $X(4)$ meets L at point K. The line through K and center of the hyperbola meets the hyperbola at F_+, F_-. Let I_+, I_-, G be the isogonal conjugate of F_+, F_- and K respectively. Let F be the inverse point of G with respect to the circumcircle of ABC. Then five points I_+, I_-, G, $X(110)$, F lie on a circle. Furthermore K lie on the Jerabek hyperbola.*

Remark: *Let the rectangular circumhyperbola is the Kiepert hyperbola of triangle ABC, theorem 2 is Parry circle theorem, you can see Parry circle in* [7], [4].

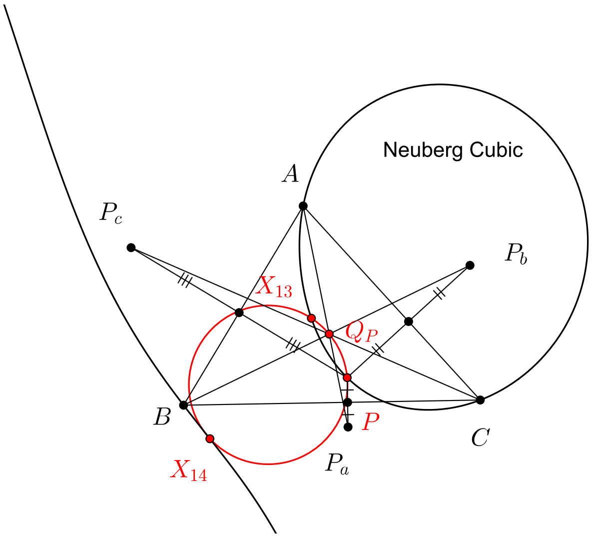

FIGURE 1. A generalization of the Lester circle

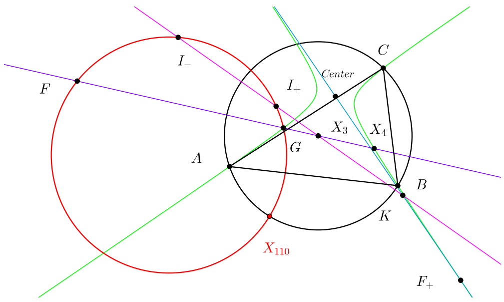

FIGURE 2. A generalization of the Parry circle

Problem 3 ([8]-[9], A generalization of the Tucker circle theorem and the Thomsen theorem associated with a conic)**.** *Let $A_1A_2A_3A_4A_5A_6$ be a hexagon, L be a line on the plane. Let L meets A_1A_2, A_2A_3, A_3A_4, A_4A_5, A_5A_6, A_6A_1 at B_2, B_3, B_4, B_5, B_6, B_1 respectively.*

Let C_1 be a point on the line A_1A_4. Let C_1B_2 meets A_2A_5 at C_2. Let C_2B_3 meets A_3A_6 at C_3. Let C_3B_4 meets A_1A_4 at C_4. Let C_4B_5 meets B_2B_5 at C_5. Let C_5B_6 meets A_3A_6 at C_6. Let C_6B_1 meets A_1A_4 at C_7. Then:

1. Six points C_1, C_2, C_3, C_4, C_5, C_6 lie on a conic if only if six points A_1, A_2, A_3, A_4, A_5, A_6 lie on a conic.

2. $C_7 = C_1$ if only if six points A_1, A_2, A_3, A_4, A_5, A_6 lie on a conic.

3. If $C_7 = C_1$ then the Pascal line of hexagon $A_1A_2A_3A_4A_5A_6$ and $C_1C_2C_3C_4C_5C_6$ and L are concurrent.

Remark *When the conic through $A_1, A_2, A_3, A_4, A_5, A_6$ is the circumcircle and L at infinity, the item 1 is the Tucker circle teorem, you can see the Tucker theorem in* [10]*. When the conic through $A_1, A_2, A_3, A_4, A_5, A_6$ is the Steiner inellipse and L at infinity, item 2 is the Thomsen theorem, you can see Thomsen theorem in* [11]*.*

Problem 4 (Another generalization Tucker circle associated with a cyclic hexagon)**.** *Let $ABCDEF$ be a cyclic hexagon. Let A_1 be any point on AD, the circle (A_1AB)*

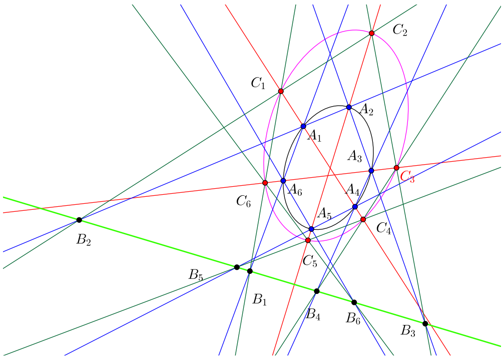

Figure 3. A generalization of the Tucker circle

meets BE again at B_1. The circle (B_1BC) meets CF again at C_1. The circle (C_1CD) meets AD again at D_1. The circle (D_1DE) meets BE again at E_1. The circle (E_1EF) meets CF again at F_1. Then show that F_1, F, A, A_1 lie on a circle and six points $A_1, B_1, C_1, D_1, E_1, F_1$ lie on a circle.

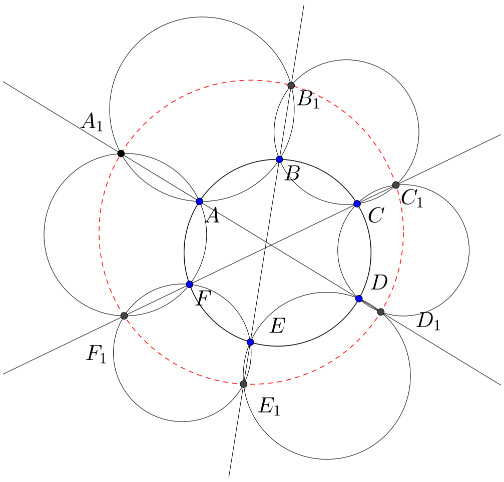

Figure 4. Another generalization of the Tucker circle

Remark: *When $B \equiv A, D \equiv C, F \equiv E$ then $A_1B_1C_1D_1E_1F_1$ be a Tucker hexagon of triangle ACE*

Problem 5. *Notation as problem 4. Let $A_2B_2C_2D_2E_2F_2$ be a hexagon, such that the sidelines of $A_2B_2C_2D_2E_2F_2$ parallel to the sidelines of $A_1B_1C_1D_1E_1F_1$ respectively. Let line A_2B_2 meets the circle (ABB_1A_1) at two points A_b, B_a. Define B_c, C_b, C_d, D_c, D_e, E_d, E_f, F_e, F_a, A_f cyclically. Then show that twelve points: A_b, B_a, B_c, C_b, C_d, D_c, D_e, E_d, E_f, F_e, F_a, A_f lie on a circle.*

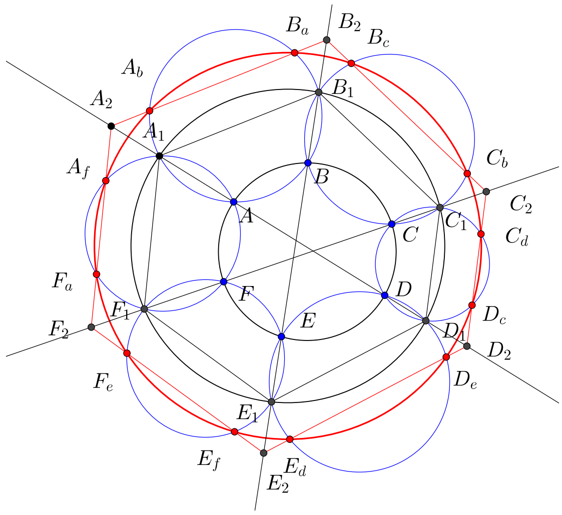

Figure 5. 12 points circle

Problem 6 ([12], A generalization of the Gauss-Bodenmiller line and the Miquel circle)**.** *Let ABC be a triangle, Let P_1 be any point on the plane. Let a line L meets BC, CA, AB at A_0, B_0, C_0 respectively. Let A_1 be a point on the plane such that B_0A_1 parallel to CP, C_0A_1 parallel to BP. Define B_1, C_1 cyclically. Define A_2, B_2, C_2, P_2 be the isogonal conjugates A_1, B_1, C_1, P respect to AB_0C_0, BC_0A_0, CA_0B_0 and ABC respectively. Let AP_2, BP_2, CP_2 meet the circumcircles of AB_0C_0, BC_0A_0, CA_0B_0 again at A_P, B_P, C_P respectively. Then show that:*

1. AA_2, BB_2, CC_2 are concurrent. Let the point of concurrence is Q

2. (A generalization of Gauss-Bodenmiller line). Four points A_1, B_1, C_1, P are collinear.

3. (A generalization of Miquel circle). Nine points A_2, B_2, C_2, P_2, A_P, B_P, C_P, Q and the Miquel point lie on a circle.

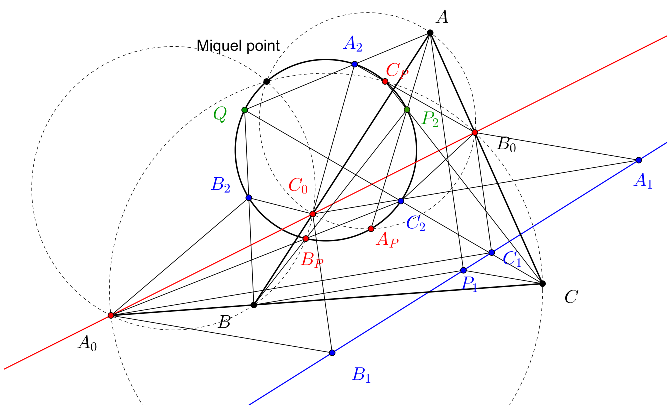

Figure 6. A generalization of the Gauss-Bodenmiller line and the Miquel circle

Remark: *When P is the orthocenter of ABC the line through A_1, B_1, C_1, P is the Gauss-Bodenmiller line ([14]-p. 172), and circles through A_2, B_2, C_2, P_2 is the Miquel circle ([14]-p.139)*

Problem 7 (A generalization of the first Droz-Farny circle)**.** *Let ABC be a triangle with circumcenter O, and the medial triangle $M_aM_bM_c$. Let O_a, O_b, O_c be three* *points on three lines OA, OB, OC respectively, such that $OO_a = OO_b = OO_c$. Let two points A_1, A_2 on the line M_bM_c; B_1, B_2 on the line M_cM_a, C_1, C_2 on the line M_aM_b, such that: $O_aA_1 = O_aA_2 = O_bB_1 = O_bB_2 = O_cC_1 = O_cC_2$. Then six points A_1, A_2, B_1, B_2, C_1, C_2 lie on a circle.*

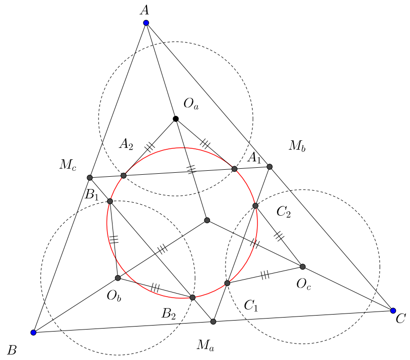

FIGURE 7. A generalization of the first Droz-Farny circle

Remark: *When $O_a \equiv A$, $O_b \equiv B$, $O_c \equiv C$, the New circle is the first Droz-Farny circle*

Problem 8 ([15], The Dual of the Maxwell theorem)**.** *Let line L be a transversal in $\triangle ABC$ crossing the sides BC, AC, AB at A_1, B_1, C_1, respectively. Let $A'B'C'$ be a triangle in the same plane, with $B'C'$, $C'A'$, $A'B'$ parallel to AA_1, BB_1, CC_1, respectively. Then three lines through A', B', C' parallel to BC, CA, AB meets $B'C', C'A', A'B'$, respectively , at three collinear points.*

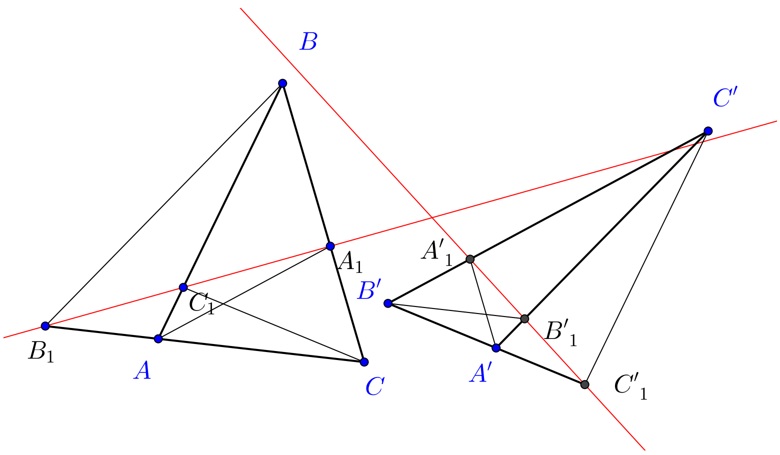

FIGURE 8. The Dual of the Maxwell theorem

Problem 9 ([16], A generalization of the Simson line)**.** *Let ABC be a triangle, let a line L through circumecenter, let a point P lie on the circumcircle. Let AP, BP, CP meets L at A_P, B_P, C_P. Denote A_0, B_0, C_0 are projection of A_P, B_P, C_P to BC , CA, AB respectively. Then A_0, B_0, C_0 and the midpoint of HP are collinear.*

Remark: *When L through P, the line $\overline{A_0B_0C_0}$ is the famous Simson line.*

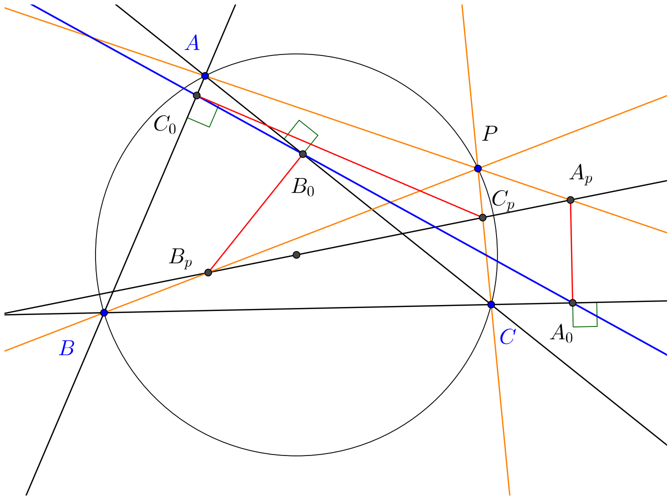

FIGURE 9. A generalization of the Simson line

Problem 10 ([17], A generalization of the Newton line)**.** *Let ABC be a triangle, and L be a line on the plane. Let $A_0B_0C_0$ be cevian triangle of a point P. Let a line meets three sidelines BC, CA, AB at A_1, B_1, C_1 respectively. The line AA_1, BB_1, CC_1 meet three sidelines B_0C_0, C_0A_0, A_0B_0 at A_2, B_2, C_2 respectively. Then three points A_2, B_2, C_2 collinear points.*

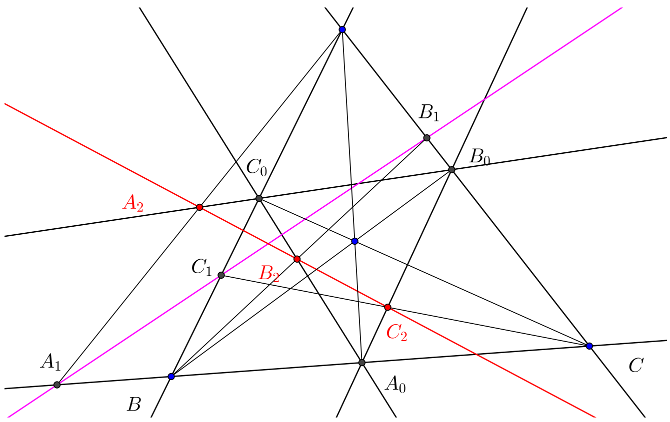

FIGURE 10. A generalization of the Newton line

Remark: *When P is the centroid of ABC, the line $\overline{A_0B_0C_0}$ is the Newton line of the quadrilateral form by (AB, BC, CA, L)*

Problem 11. *Let ABC be a triangle, let (S) be a circumconic of ABC, let P be a point on the plane. Let the lines AP, BP, CP meet the conic again at A', B', C'. Let P be a point on the plane. Let $A_0 = DA' \cap BC$, $B_0 = DB' \cap AC$; $C_0 = DC' \cap AB$. Let AP meets BC at P_a.*

1. If D be a point on the polar of point P with respect to (S), then A_0, B_0, C_0 are collinear [22].

2. If D lies on the conic (S). Let the line through D and parallel to AP, this line meets BC at D_a. Let D'_a on the ray DD_a such that $\frac{\overline{DD_a}}{\overline{DD'_a}} = \frac{\overline{A'P_a}}{\overline{A'P}}$. Define D'_b, D'_c cyclically. Then seven points $A_0, B_0, C_0, D'_a, D'_b, D'_c$ and P are collinear.

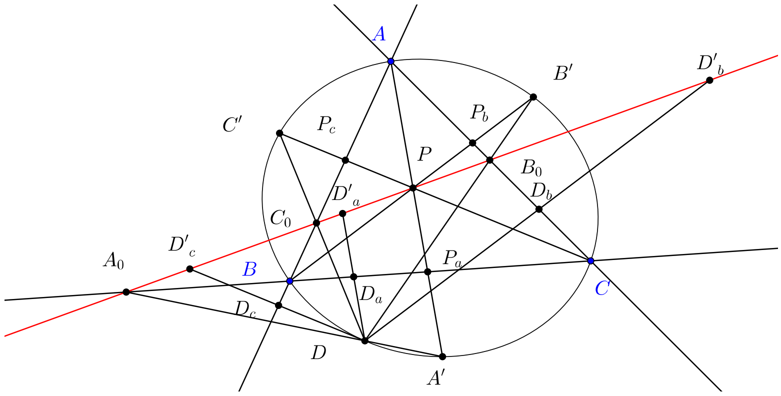

FIGURE 11. A new theorem associated with a conic and pole-polar.

Remark 1. *When P is the orthocenter of ABC, and (S) is the circumcircle of ABC, then the line $\overline{D'_aD'_bD'_c}$ is the Steiner line* [18]. *You can see a proof of statement A_2, B_2, C_2, P collinear in (*[19]-[22]*). But D'_a, D'_b, D'_c lie on the line $\overline{A_2B_2C_2P}$, have not been proven.*

Problem 12 (A generalization of the Van Aubel theorem-[23])**.** *Let $ABCD$ be a quadrilateral, let four points A_1, B_1, C_1, D_1 on the plane, such that: $\angle A_1AB = \angle DAD_1 = \alpha$; $\angle B_1BC = \angle ABA_1 = \beta$; $\angle BCB_1 = \angle C_1CD = \gamma$; $\angle CDC_1 = \angle D_1DA = \delta$; and $\alpha + \gamma = \beta + \delta = \frac{\pi}{2}$ then $A_1B_1C_1D_1$ is an orthodiagonal quadrilateral.*

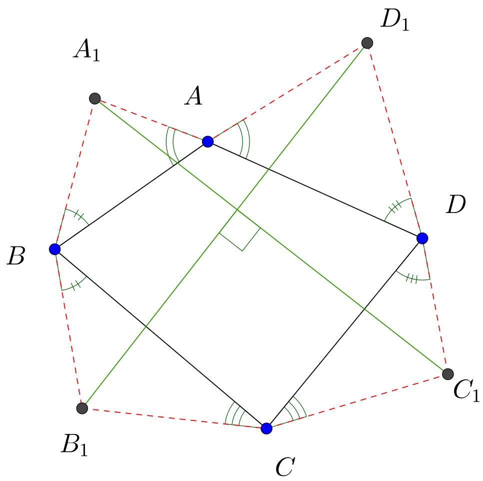

FIGURE 12. A generalization of the Van Aubel theorem

Remark: *When $\alpha = \gamma = \beta = \delta = \frac{\pi}{4}$, the problem 13 is Van Aubel theorem* [24]

Problem 13 ([25])**.**

1. Let 2n-convex cyclic polygon $A_1A_2A_3...A_{2n}$, let P be a point in the Euclidean three-space, then:

$$\sum_{i=1}^{2n}(-1)^{i+1}.PA_i^2.\frac{A_{i-1}A_{i+1}}{A_iA_{i-1}.A_iA_{i+1}}=0 \tag{1}$$

Where $A_0=A_{2n}$ and $A_{2n+1}=A_1$

2. Let two direct similar 2n-convex cyclic polygon $A_1A_2A_3...A_{2n}$ and $B_1B_2B_3...B_{2n}$, then:

$$\sum_{i=1}^{2n}(-1)^{i+1}.B_iA_i^2.\frac{A_{i-1}A_{i+1}}{A_iA_{i-1}.A_iA_{i+1}}=0 \tag{2}$$

Where $A_0=A_{2n}$ and $A_{2n+1}=A_1$ and $B_0=B_{2n}$ and $B_{2n+1}=B_1$

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