TO TEACH DEFINITIONS IN GEOMETRY OR TEACH TO DEFINE?

Michael de Villiers, University of Durban-Westville, South Africa

This paper argues from a theoretical standpoint that students should be actively engaged in the defining of geometric concepts like the quadrilaterals, and presents some data relating to a teaching experiment aimed at developing students' ability to define.

Introduction

Already early in this century the German mathematician Felix Klein (1924) came out strongly against the practice of presenting mathematical topics as completed axiomatic-deductive systems, and instead argued for the use of the so-called "*bio-genetic*" principle in teaching. The genetic approach has also been advocated by Wittmann (1973), Polya (1981), Freudenthal (1973) and many others. Essentially, the genetic approach departs from the standpoint that the learner should either retrace (at least in part) the path followed by the original discoverers or inventors, or to retrace a path by which it could have been discovered or invented. In other words, learners should be exposed to or engaged with the typical mathematical processes by which new content in mathematics is discovered, invented and organized. Human (1978:20) calls it the "*reconstructive*" approach and contrasts it as follows with the so-called "*direct axiomatic-deductive*" approach:

> "*With this term we want to indicate that content is not directly introduced to pupils (as finished products of mathematical activity), but that the content is newly reconstructed during teaching in a typical mathematical manner by the teacher and/or the pupils*." (freely translated from Afrikaans)

The didactical motivation for the reconstructive approach includes, among others, the following elements, namely, that its implementation highlights the *meaning* (actuality) of the content, and that it allows students to *actively participate* in the construction and the development of the content. With different content (definitions, axiom systems, propositions, proofs, algorithms, etc.) one can of course distinguish different mathematical processes by which that content can be constructed (eg. defining, axiomatizing, conjecturing, proving, algorithmatizing, etc.). A genetic or reconstructive approach is therefore characterized by not presenting content as a finished (prefabricated) product, but rather to focus on the genuine mathematical processes by which the content can be developed or reconstructed. Note however that a reconstructive approach does not necessarily imply learning by discovery for it may just be a reconstructive explanation by the teacher or the textbook.

Defining

The direct teaching of geometry definitions with no emphasis on the underlying process of defining has often been criticised by mathematicians and mathematics educators alike. For example, already in 1908 Benchara Blandford wrote (quoted in Griffiths & Howson, 1974: 216-217):

> "*To me it appears a radically vicious method, certainly in geometry, if not in other subjects, to supply a child with ready-made definitions, to be subsequently memorized after being more or less carefully explained. To do this is surely to throw away deliberately one of the most valuable agents of intellectual discipline. The evolving of a workable definition by the child's own activity stimulated by appropriate questions, is both interesting and highly educational.*"

In A. Olivier & K. Newstead (Eds), **Proceedings of the Twenty-second International Conference for the Psychology of Mathematics Education: Vol. 2.** (pp. 248-255). University of Stellenbosch: Stellenbosch, 12-17 July 1998.

The well-known mathematician Hans Freudenthal (1973:417-418) also strongly criticized the traditional practice of the direct provision of geometry definitions claiming that most definitions are not preconceived, but the finishing touch of the organizing activity, and that the child should not be denied this privilege. Ohtani (1996:81) has argued that the traditional practice of simply telling definitions to students is a method of moral persuasion with several social functions, amongst which are: to justify the teacher's control over the students; to attain a degree of uniformity; to avoid having to deal with students' ideas; and to circumvent problematic interactions with students. Vinner (1991) and many others have presented arguments and empirical data that just knowing the definition of a concept does not at all guarantee understanding of the concept. For example, although a student may have been taught, and be able to recite, the standard definition of a parallelogram as a quadrilateral with opposite sides parallel, the student may still not consider rectangles, squares and rhombi as parallelograms, since the students' concept image of a parallelogram is one in which not all angles or sides are allowed to be equal.

Linchevski, Vinner & Karsenty (1992) have further reported that many student teachers do not even understand that definitions in geometry have to be economical (contain no superfluous information) and that they are arbitrary (in the sense, that several alternative definitions may exist). It is plausible to conjecture that this is probably due to their past school experiences where definitions were probably supplied directly to them. It would appear that in order to increase students' understanding of geometric definitions, and of the concepts to which they relate, it is essential to engage them at some stage in the process of defining of geometric concepts. Due to the inherent complexity of the process of defining, it would also appear to be unreasonable to expect students to immediately come up with formal definitions on their own, unless they have been guided in a didactic fashion through some examples of the process of defining which they can later use as models for their own attempts. Furthermore, the construction of definitions (defining) is a mathematical activity of no less importance than other processes such as solving problems, making conjectures, generalizing, specializing, proving, etc., and it is therefore strange that it has been neglected in most mathematics teaching. In mathematics we can distinguish between two different types of defining of concepts, namely, **descriptive** (*a posteriori)* and **constructive** (*a priori*) defining (e.g. compare Krygowska, 1971; Human, 1978:164-165; De Villiers, 1986;1994).

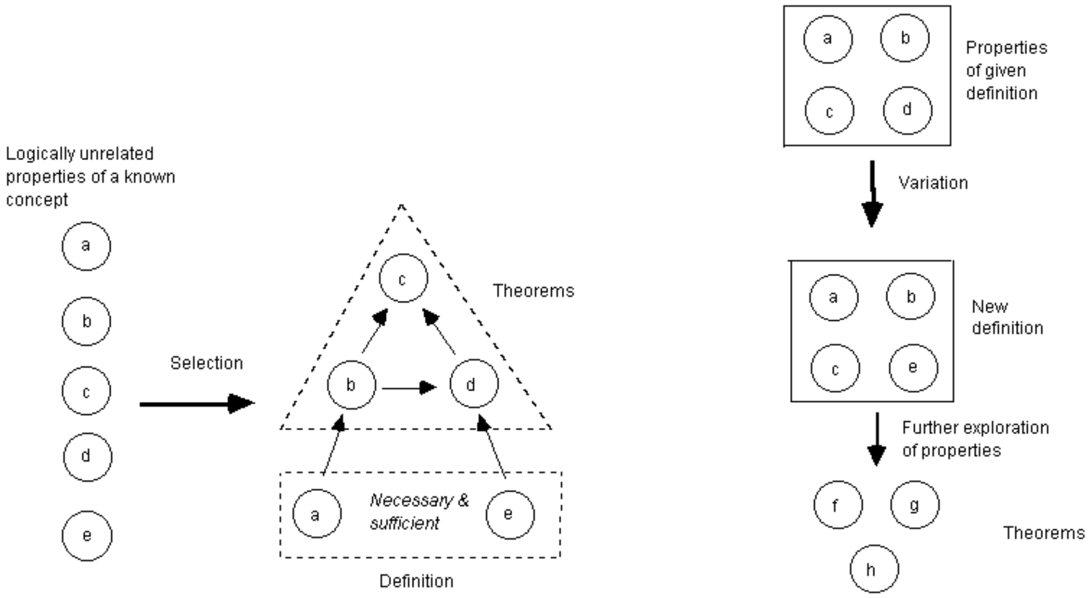


Figure 1

Descriptive defining

"... the describing definition ... outlines a known object by singling out a few characteristic properties". - Hans Freudenthal (1973 : 458)

With the descriptive (*a posteriori*) defining of a concept is meant here that the concept and its properties have already been known for some time and is defined only afterwards (see Figure 1a). *A posteriori* defining is usually accomplished by selecting an appropriate subset of the total set of properties of the concept from which all the other properties can be deduced. This subset then serves as the definition and the other remaining properties are then logically derived from it as theorems.

Constructive defining

"... the algorithmically constructive and creative definition ... models new objects out of familiar ones" - Hans Freudenthal (1973 : 458).

Constructive *(a priori)* defining takes places when a given definition of a concept is changed through the exclusion, generalization, specialization, replacement or addition of properties to the definition, so that a new concept is constructed in the process (see Figure 1b). In other words, a new concept is defined "*into being*", the further properties of which can then be experimentally or logically explored. Whereas the main purpose or function of *a posteriori* defining is that of the ***systematization*** of existing knowledge, the main function of *a priori* defining is the production of ***new*** knowledge. We shall further on mainly focus on a discussion of the teaching and learning of the process of descriptive defining.

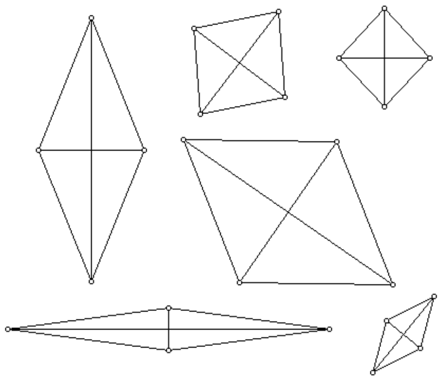
The USEME experiment

From the Van Hiele theory, it is clear that understanding of formal definitions can only develop at Level 3, since that is where students start noticing the inter-relationships between the properties of a figure. Is it possible to devise teaching strategies for the learning of the process of defining at Van Hiele Level 3? This in fact was the focus of the University of Stellenbosch Experiment with Mathematics Education (USEME) conducted with a control group in 1977 and an experimental group in 1978 (see Human & Nel et al, 1989a). The experiment was aimed at the Grade 10 (Std 8) level and involved 19 schools in the Cape Province. Whereas the traditional approach focusses overridingly on developing the ability of making deductive proofs (especially for riders), the experimental approach was (among others) aimed mainly at:

- letting students realize: (1) that different, alternative definitions for the same concept are possible; (2) that definitions may be uneconomical or economical; (3) that some economical definitions lead to shorter, easier proofs of properties
- developing students' ability to construct formal, economical definitions for geometrical concepts

The following is an example of one of the first exercises in (descriptive) defining used in the experimental approach (see Human & Nel et al, 1989b:21). Note that although these students had already come across the concept "*rhombus*", they had not been given any definition in earlier classes.

EXERCISE



1(a) Make a list of all the common properties of the figures above. Look at the angles, sides and diagonals and measure if necessary.

(b) What are these types of quadrilaterals called?

(c) How would you explain in words, *without making a sketch*, what these quadrilaterals are to someone not yet acquainted with them?

The spontaneous tendency of almost all the students in (c) was to make a list of all the properties discovered and listed in (a); thus giving a correct, but uneconomical description (definition) of the rhombi (thus suggesting Van Hiele Level 2 understanding). This led to the next two exercises which were intended to lead them to shorten their descriptions (definitions) by considering leaving out some properties.

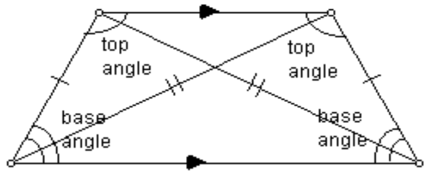
Typically the students then came up with different shorter versions, some of which were *incomplete* (particularly if they're encouraged to make them as short as possible by promising a prize!), for example: "*A rhombus is a quadrilateral with perpendicular diagonals*". This provided opportunity to provide a counter-example and a discussion of the need to contain enough (sufficient) information in one's descriptions (definitions) to ensure that somebody else knows exactly what figure one is talking about. Also note at this stage that they were not expected to **logically** check their definitions, but expected to check whether the conditions contained in their definitions provided sufficient information for the accurate **construction** of a rhombus.

Psychologically, constructions like these are extremely important for the transition from Van Hiele Level 2 to Level 3, since it helps to develop an understanding of the logical structure of "*if-then*" statements (compare Smith, 1940). For example, students learned to distinguish clearly between the relationships they **put** into a figure (the premisse) and the relationships which **resulted** without any action on their part (the conclusion).

The students were then led into a deductive phase where starting from one definition they had to logically check whether all the other properties could be derived from it (as theorems). The same exercises were then repeated for the parallelograms. Eventually, it was explained to students that it would be confusing if everyone used different definitions for the rhombi and parallelograms, and it was agreed to henceforth use one definition only for each concept.

In order to evaluate whether students had developed some ability to formally define geometric concepts themselves, the following were some of the questions given afterwards to the experimental, as well as the control group. The first question was of a known concept that both groups had already treated in class (the control group in a direct way & the experimental group in a reconstructive way). So essentially they just needed to recall a definition done in class. This question therefore served only as a base line against which to judge their ability to define in the next question which was of a completely new concept that had been not treated at all in any of the groups.

1. Give a definition of the parallelograms.
2. Quadrilaterals which look like the one below is called a *regular trapezium*.



The regular trapeziums have among others the following properties:

(1) One pair of opposite sides parallel, but not equal.

(2) Diagonals are equal.

(3) Base angles are equal (see figure).

(4) Top angles are equal (see figure).

(5) A top angle and base angle are togther equal to 180°.

(6) One pair of opposite sides are equal, but not parallel.

Answer the following questions:

(a) Provide a definition (as short as possible) of the regular trapeziums.

(b) Prove that the properties of regular trapeziums not mentioned in your definition, indeed logically follow from your definition.

Table 1 gives the results that were obtained. Note that both groups had the same teachers and that they were statistically comparable in terms of IQ, language ability, etc. It is immediately noticeable that the experimental group gave higher percentages of correct, economical definitions in both cases. The experimental group also gave fewer correct, uneconomical definitions in both cases. This improvement in terms of economy of definition for the experimental group, however, appeared to be at a slight cost in relation to Question 1, in the sense that there was a slightly higher number of faulty definitions which contained insufficient properties. It is possible that this increase was due to uncritical attempts at producing economical definitions. This indicates a possible risk of the experimental approach. What was perhaps extremely surprising was that both the control and experimental groups performed better in defining the unknown concept than the known concept. A possible explanation could be that in Question 2, the act of constructing a definition themselves, forced them to more carefully consider the underlying logical relationships, than to just uncritically try and recall a previously learnt definition in Question 1.

		Question 1	**Question 2**
Correct economical	Control	25%	44%
	Experimental	54%	58%
Faulty	Control	22%	8%
	Experimental	26%	4%
Correct uneconomical	Control	51%	47%
	Experimental	19%	39%
None	Control	2%	0%
	Experimental	0%	0%

Table 1

Further discussion

From the constructivist assumption that meaningful knowledge needs to be actively (re)-constructed by the learner, it also follows that students should be engaged in the activity of defining and allowed to choose their own definitions at each Van Hiele level. This implies allowing the following kinds of meaningful definitions at each Van Hiele level (compare Burger & Shaughnessy, 1986):

Van Hiele 1: *Visual* definitions, eg. a rectangle is a quad that looks like this (draws or identfies one) or describes it in terms of *visual* properties, eg. all angles 90°, two long and two short sides.

Van Hiele 2: *Uneconomical* definitions, eg. a rectangle is a quadrilateral with opposite sides parallel and equal, all angles 90°, equal diagonals, half-turn-symmetry, two axes of symmetry through opposite sides, two long and two short sides, etc.

Van Hiele 3: *Correct, economical* definitions, eg. a rectangle is a quadrilateral with an axis of symmetry through each pair of opposite sides.

The first two examples show that students' definitions at these levels would tend to be *partitional*, in other words, they would not allow the inclusion of the squares among the rectangles (by explicitly stating two long and two short sides). In contrast, according to the Van

Hiele theory, definitions at Level 3 are typically *hierarchical*, which means they allow for the inclusion of the squares among the rectangles, and would not be understood by students at lower levels. However, research reported in De Villiers (1994) show that many students who exhibit excellent competence in logical reasoning at Level 3, if given the opportunity, still prefer to define quadrilaterals in *partitions*. (In other words, they would for example define a parallelogram as a quadrilateral with both pairs of opposite sides parallel, but not all angles or sides equal).

For this reason, students should not simply be supplied with ready-made definitions for the quadrilaterals, but allowed to formulate their own definitions irrespective of whether they are partitional or hierarchical. By then discussing and comparing in class the relative advantages and disadvantages of these two different ways of classifying and defining quadrilaterals (both of which are mathematically correct), students may be led to realize that there are certain advantages in accepting a hierarchical classification (compare De Villiers, 1994). For example, if students are asked to compare the following two definitions for the parallellograms, they immediately realize that the former is much more **economical** than the latter:

> *hierarchical:* A parallelogram is a quadrilateral with both pairs of opposite sides parallel.
>
> *partitional:* A parallelogram is a quadrilateral with both pairs of opposite sides parallel, but not all angles or sides equal.

Clearly in general, partitional definitions are longer since they have to include additional properties to ensure the exclusion of special cases. Another advantage of a hierarchical definition for a concept is that all theorems proved for that concept then automatically apply to its special cases. For example, if we prove that the diagonals of a parallelogram bisect each other, we can immediately conclude that it is also true for rectangles, rhombi and squares. If however, we classified and defined them partitionally, we would have to prove separately in each case, for parallelograms, rectangles, rhombi and squares, that their diagonals bisect each other. Clearly this is very uneconomical. It seems clear that unless the role and function of a hierarchical classification is meaningfully discussed in class, many students will have difficulty in understanding why their own partitional definitions are not used.

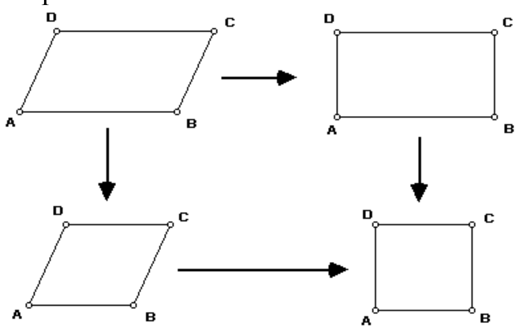


Figure 2

On the other hand, the dynamic nature of geometric figures constructed in *Sketchpad* or *Cabri* may also make the acceptance of a hierachical classification of the quadrilaterals far easier. For example, if students construct a quadrilateral with opposite sides parallel, then they will notice that they could easily drag it into the shape of a rectangle, rhombus or square as shown in Figure 2. (Recently in a session on *Sketchpad* with my 8-year old son, he had no difficulty dragging a parallelogram into the shape of a square and a rectangle, and then accepting that they were special cases). In fact, it seems quite possible that with dynamic software, students would be able to accept and understand this even at Van Hiele Level 1 (Visualization), but further research into this particular area is needed.

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