Slightly modified from Meyer, JH & Van Biljon, A, (2009), *Proceedings of the 15th Annual AMESA Congress*, University of Free State, Bloemfontein, pp. 1-8.

From the Fermat point to the De Villiers points of a triangle

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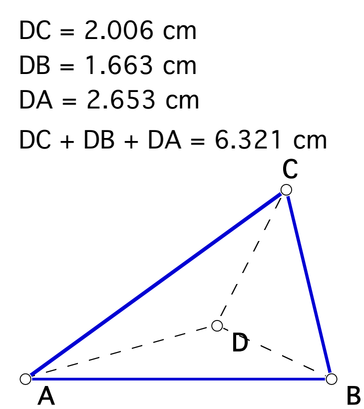
In August 2008, I accidentally found out to my great surprise that two special points of a triangle have been named the De Villiers points after me at the *WolframMathWorld* (Weisstein, no date) and that they are also referenced as Points 1127 and 1128 at the *Encyclopedia of Triangle Centers* (Kimberling, no date). But it was also immediately humbling (and bemusing) to note that there are more than 3500 special points known in relation to the simple, elementary triangle, so these are only two amongst thousands!

Be that as it may, the purpose of this paper is to provide a brief background leading up to my discovery of these points, and the proofs involved, which should be accessible and informative for talented mathematics learners and their teachers.

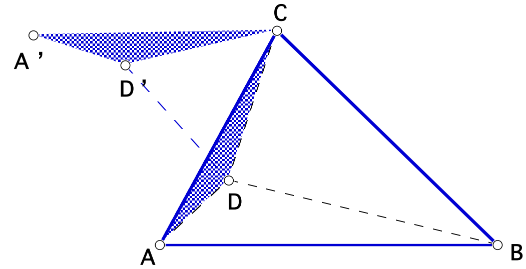
Let's start by considering the following *Sketchpad* investigation from De Villiers (2003b).



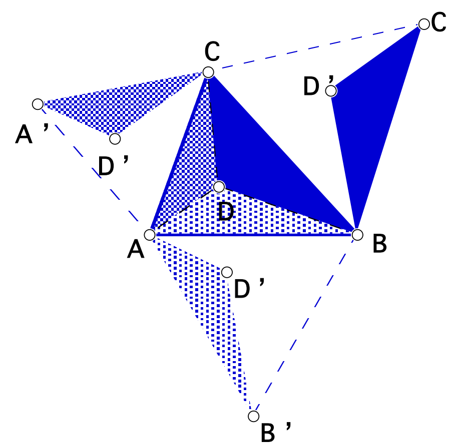
Airport Problem

Suppose an airport is planned to service three cities of more or less equal size. The planners decide to locate the airport so that the sum of the distances to the three cities is a minimum. Where should the airport be located?

Solution



Rotate triangle *ADC* by - 60° around point *C* to get triangle *A'D'C*. From the rotation, it follows that *CD* = *CD'*, and since angle *D'CD* measures 60°, it follows that triangle *DCD'* is equilateral. Since *AD* = *A'D'* from the rotation, we now have *AD* + *CD* + *BD*



= $A'D' + D'D + DB$. But the path from A' to B (e.g. $A'D' + D'D + DB$) will be a minimum when it is straight, in which case, angle $A'D'C = 120°$, and therefore angle $ADC = 120°$. From symmetry it follows that the other two angles around D will also be equal to 120°. Thus, the solution of the problem is to place the airport where these three angles around D all equal 120°. It is now not hard to see that D can be located simply by constructing equilateral triangles $A'AC$, $B'BA$ and $C'CB$ on the sides of triangle ABC (see the diagram at right) and constructing the straight lines $A'B$, $B'C$ and $C'A$ to meet at D.

Figure 1: Fermat and Torricelli

Historical Notes

The point D is usually called the inner Fermat point[1] of a triangle after Pierre de Fermat who first posed the problem in the 1600s of finding a point inside an *acute* triangle so that the sum of the distances to the vertices is a minimum.[2] However, more correctly, it should probably be called the Fermat-Torricelli point as the Italian mathematician and scientist Evangelista Torricelli was the first to solve the problem and propose constructing equilateral triangles on the sides to locate the optimal point. Of some cultural-historical interest is that the Italian and French mathematical communities are apparently still arguing about who the point should be named after! The transformation

[1] The outer Fermat point is obtained by constructing the equilateral triangles inwardly, then similarly drawing the concurrent lines $A'B$, $B'C$ and $C'A$.

[2] When one of the angles of the triangle is 120° or greater, then the Fermat point (which still exists) is no longer the point that minimizes the sum of the distances to the vertices, but the minimal point is located at the vertex of the obtuse angle.

proof given above was more recently invented in 1929 by the German mathematician J. Hoffman.

The centroid of a triangle

The following fundamental geometry result still appears in some high school geometry texts, but unfortunately mostly without proof: *"The three medians of any triangle are concurrent at the centroid*." Let's consider a non-traditional proof based on areas, but that will give us further insight leading to an interesting, and important generalization.

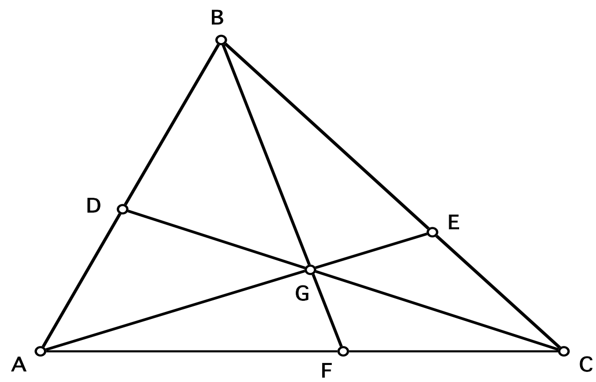


Figure 1: Centroid of triangle

Proof

1. Let AE and CD be medians intersecting at point G as shown in Figure 1. Join B with G and extend to F on AC. We now have to show that F is the midpoint of AC. (In other words, that BF is also a median and therefore that all three meet in the same point G.)
2. If we denote the area of a triangle by the following notation, area $\Delta ABC \leftrightarrow (ABC)$, we have:

   $$\frac{(BAF)}{(BFC)} = \frac{\frac{1}{2}h_1 AF}{\frac{1}{2}h_1 FC} = \frac{AF}{FC} \text{ and } \frac{(GAF)}{(GFC)} = \frac{\frac{1}{2}h_2 AF}{\frac{1}{2}h_2 FC} = \frac{AF}{FC}.$$

   Therefore: $\frac{AF}{FC} = \frac{(BAF)}{(BFC)} = \frac{(GAF)}{(GFC)} = \frac{(BAF)-(GAF)}{(BFC)-(GFC)} = \frac{(BAG)}{(BCG)}$ ... dividendo.

   Similarly, we find: $\frac{CE}{EB} = \frac{(ACG)}{(BAG)}$ and $\frac{BD}{DA} = \frac{(BCG)}{(ACG)}$.
3. But it is given that $BE = EC$ and $BD = DA$. Therefore, $(BCG) = (ACG)$ and $(ACG) = (BAG)$ which implies $(BAG) = (BCG)$. But the areas of these two triangles are proportional to AF and FC as shown by the second equation. Thus, $AF/FC = 1$ implies $AF = FC$ and completes the proof.

Looking back: Ceva's Theorem

Now look back carefully at the proof. Only consider the product of the three ratios AF/FC, CE/EB and BD/DA expressed in terms of areas in Step 2. What do you notice about this product, and whether in deriving these three ratios, the properties that *E* and *D* are midpoints were used at all? What can we therefore conclude from this?

Note that:
$\frac{AF}{FC}\times\frac{CE}{EB}\times\frac{BD}{DA}=\frac{(BAG)}{(BCG)}\cdot\frac{(ACG)}{(BAG)}\cdot\frac{(BCG)}{(ACG)}=1$. More over, the properties that *E* and *D* are midpoints were not used at all in this derivation! Therefore we can immediately generalize, e.g. if in any triangle, line segments *AE*, *BF* and *CD* are concurrent (with *E*, *F* and *D* respectively on sides *BC*, *AC* and *AB*) then $\frac{AF}{FC}\times\frac{CE}{EB}\times\frac{BD}{DA}=1$. The converse of this result is also true, and can be proved by using proof by contradiction.[3]

This interesting, major result is called Ceva's Theorem after an Italian mathematician named Giovanni Ceva (1648-1734) who published his theorem in 1678 and proving it by considering centres of gravity and the law of moments when different point masses are placed at the vertices of a triangle.[4] In his honour the line segments *AE*, *BF* and *CD* joining the vertices of a triangle to any given points on the opposite sides, are called *cevians*. (Note that apart from the medians, the altitudes and angle bisectors of a triangle can be considered as cevians if extended to meet the opposite sides). The converse of Ceva's theorem is a powerful theorem for proving various concurrencies of lines, and all learners preparing for the 3rd round of the South African Mathematics Olympiad should know it.

Generalizing the Fermat-Torricelli point

The Fermat-Torricelli point can be generalized further by respectively placing a) congruent, b) similar isosceles or c) similar triangles on the sides, but all are special cases of the following unifying generalization from De Villiers (1995):

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[3] Pedagogically, the above example can in a teaching situation be used to illustrate the *discovery* function of proof as mentioned in De Villiers (2003a), whereby sometimes proving a result and reflecting on the proof carefully in the style of Polya, can lead to a further generalization.

[4] An online physical simulation of Ceva's theorem, showing how to experimentally find the centroid of a triangle with different point masses at the vertices is available at http://math.kennesaw.edu/~mdevilli/finding-centroid-ceva.html

"If triangles *DBA*, *ECB* and *FAC* are constructed outwardly (or inwardly) on the sides of any ΔABC so that $\angle DAB = \angle CAF$, $\angle DBA = \angle CBE$ and $\angle ECB = \angle ACF$ then *DC*, *EA* and *FB* are concurrent."

In order to prove this result we will use the following lemma, which is stated here without proof.

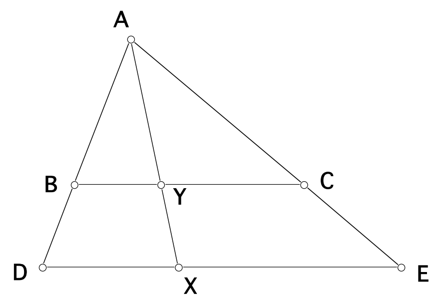

Figure 2: Lemma

Lemma

Triangle *ABC* is given. Extend *AB* and *AC* to *D* and *E* respectively so that *DE*//*BC*. Choose any point *Y* on *BC* and extend *AY* to *X* on *DE* (see Figure 2). Then *BY*/*YC* = *DX*/*XE*.

Proof of the Fermat-Torricelli generalization

Assume that the lines we want to prove concurrent intersect *BC*, *CA* and *AB* respectively at *X*, *Y* and *Z*. Extend *AB* to *G* and *AC* to *H* so that *GEH*//*BC* (see Figure 3). Label *BE*, *EC*, *CF*, *FA*, *AD* and *DB* respectively as s_1, s_2, s_3, s_4, s_5 and s_6. Then $\angle BGE = \angle ABC$ and $\angle BEG = b$.

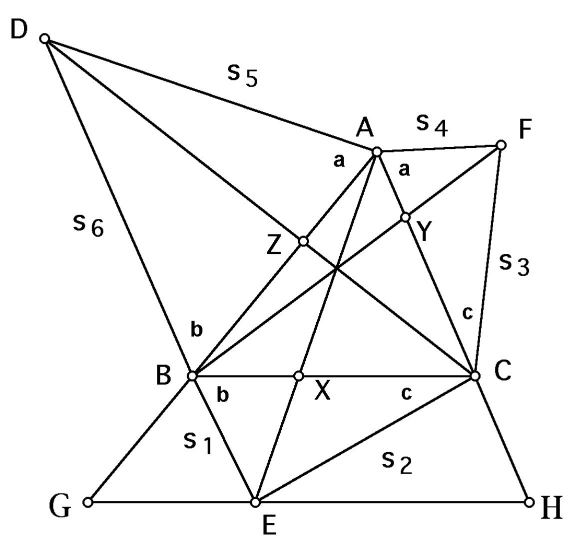

Figure 3: Proof of Fermat-Torricelli generalization

According to the sine rule:

$$\frac{GE}{\sin(\angle GBE)} = \frac{s_1}{\sin(\angle ABC)}$$

$$\frac{GE}{\sin(b + \angle ABC)} = \frac{s_1}{\sin(\angle ABC)}$$

$$GE = \frac{s_1 \sin(b + \angle ABC)}{\sin(\angle ABC)}$$

Similarly we obtain

$$EH = \frac{s_2 \sin(c + \angle ACB)}{\sin(\angle ACB)}.$$

According to the preceding Lemma therefore

$$\frac{BX}{XC} = \frac{GE}{EH} = \frac{s_1 \sin(b + \angle ABC)}{\sin(\angle ABC)} \cdot \frac{\sin(\angle ACB)}{s_2 \sin(c + \angle ACB)}.$$

In the same way we have

$$\frac{CY}{YA} = \frac{s_3 \sin(c + \angle ACB)}{\sin(\angle ACB)} \cdot \frac{\sin(\angle CAB)}{s_4 \sin(a + \angle CAB)}$$

$$\frac{AZ}{ZB} = \frac{s_5 \sin(a + \angle CAB)}{\sin(\angle CAB)} \cdot \frac{\sin(\angle ABC)}{s_6 \sin(b + \angle ABC)}$$

Therefore, $\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = \frac{s_1}{s_2} \cdot \frac{s_3}{s_4} \cdot \frac{s_5}{s_6}$ ... (3)

Applying the sine rule to triangles *ECB*, *FAC* and *DBA* we obtain

$$\frac{s_1}{s_2} = \frac{\sin(c)}{\sin(b)}; \frac{s_3}{s_4} = \frac{\sin(a)}{\sin(c)}; \frac{s_5}{s_6} = \frac{\sin(b)}{\sin(a)}$$

By substitution into (3) therefore $\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1$ so that *AX*, *BY* and *CZ* are concurrent according to the converse of Ceva's theorem. But then *EA*, *FB* and *DC* are also concurrent.

The generalization is not new, and the earliest proof I'm aware of is from 1936 by N. Alliston in *The Mathematical Snack Bar* by W. Hoffer, pp. 13-14. Of practical relevance is that the Fermat-Torricelli generalization can be used to solve a 'weighted' airport problem, for example, when the populations in the three cities are of different size. I was also a few months ago contacted by a mathematical biologist, Stephen Doro from Columbia University's College of Physicians and Surgeons, USA who was looking at its application in the branching of larger arteries and veins in the human body into smaller and smaller ones.

The De Villiers points of a triangle

On the basis of an often-observed (but not generally true) duality between circumcentres and incentres, I conjectured in De Villiers (1996) that the following might be true from a similar result for circumcentres (Kosnita's theorem), namely:

> The lines joining the vertices A, B, and C of a given triangle ABC with the incentres of the triangles BCO, CAO, and ABO (O is the incentre of ΔABC), respectively, are concurrent (in what is now called the inner De Villiers point).

Investigation on the dynamic geometry program *Sketchpad* quickly confirmed that the conjecture was indeed true. (For an interactive sketch online, see De Villiers, 2009). Using the aforementioned generalization of the Fermat-Torricelli point, it was now very easy to prove this result.

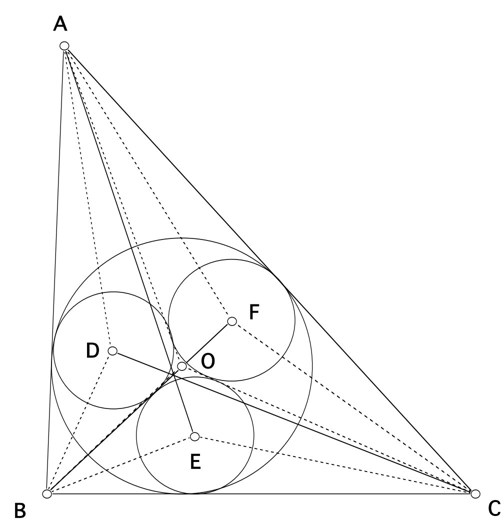

Figure 4: The inner De Villiers point

Proof

As shown in Figure 4 we have that $\angle DAB = \frac{1}{4}\angle A = \angle CAF$, $\angle DBA = \frac{1}{4}\angle B = \angle CBE$ and $\angle ECB = \frac{1}{4}\angle C = \angle ACF$, and from the Fermat-Torricelli generalization it therefore follows that DC, EA and FB are concurrent.

The outer De Villiers point is obtained when the excircles are constructed as shown in Figure 5, in which case the lines joining the vertices A, B, and C of a given triangle ABC with the incentres of the triangles BCI_1, CAI_2, and ABI_3 (I_i are the excentres of ΔABC), are concurrent. The proof follows similarly from the Fermat-Torricelli generalization.

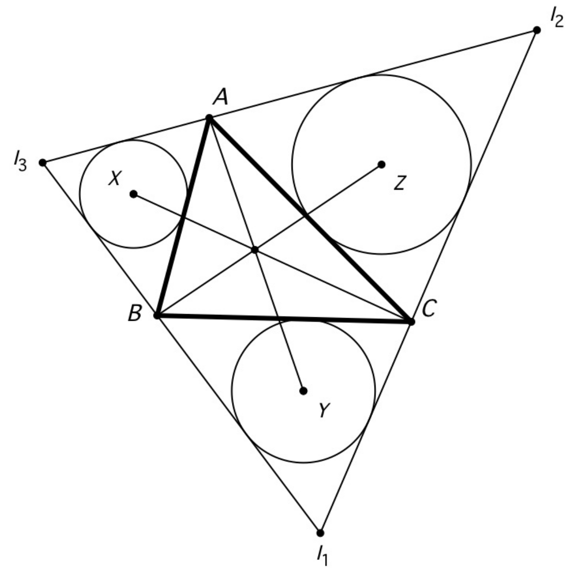

Figure 5: The outer De Villiers point

Concluding comment

Although the De Villiers points are perhaps of minimal mathematical importance, the preceding tale does illustrate a typical mathemathematical investigation that includes elements of experimentation, conjecturing, deductive verification and explanation. It also shows that Euclidean geometry is certainly not exhausted and the recent invention of dynamic geometry software has stimulated renewed interest in the subject (Davies, 1995). For example, a few years ago June Lester (1997) from Canada used *Sketchpad* to experimentally discover that the two Fermat points, nine-point centre and the circumcentre of a triangle all lie on a circle (now known as the Lester circle).

The Fermat-Torricelli generalization by itself is, however, an interesting and powerful result that deserves to be known better. Apart from the collinearity of the two De Villiers points with other special points as mentioned at the *Encyclopedia of Triangle Centers* (Kimberling, undated), I've since found that together with the vertices of the base triangle, they lie on a hyperbola, and that the first (inner) De Villiers point appears to also exist in hyperbolic geometry (see De Villiers, 2009). It is therefore possible that further interesting properties may be found.

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