which is only a different manner of writing (1), we obtain

$$\frac{1}{a_{\nu+1}} + \frac{2}{b_{\nu+1}} < \frac{1}{a_\nu} + \frac{2}{b_\nu}$$

or

$$\frac{3a_{\nu+1}b_{\nu+1}}{2a_{\nu+1} + b_{\nu+1}} > \frac{3a_\nu b_\nu}{2a_\nu + b_\nu},$$

or, in abbreviated form, if we set

$$\frac{3a_\nu b_\nu}{2a_\nu + b_\nu} = B_\nu,$$

then

$$(4) \qquad B_{\nu+1} > B_\nu.$$

The inequalities (3) and (4) imply that as ν increases, A_ν grows continuously smaller, B_ν continuously larger.

Since for infinitely great ν, both A_ν and B_ν become the circumference u of the circle, for every finite ν it must be true that

$$B_\nu < u < A_\nu.$$

The limits A_ν and B_ν of this inequality are much narrower than the Archimedes limits a_ν and b_ν. If we take the hexagon, for example, as our initial polygon and $d = 1$, then $a_0 = 2\sqrt{3}$, $b_0 = 3$, $u = \pi$, and we obtain $A_1 = 3.1423$ and $B_0 = 3.1402$; thus we are able to obtain the correct value of π to two accurate decimal places by using only the inscribed hexagon and the circumscribed dodecagon, whereas the same precision is achieved by the Archimedes method only with the use of the polygon of 96 sides.

39 Fuss' Problem of the Chord-Tangent Quadrilateral

To find the relation between the radii and the line joining the centers of the circles of circumscription and inscription of a bicentric quadrilateral.

A *bicentric* or *chord-tangent quadrilateral* is defined as a quadrilateral that is simultaneously inscribed in one circle and circumscribed about another. Let $PQRS$ be such a quadrilateral, $\mathfrak{C}$ the circumscribed circle, Γ the inscribed circle. Let the points of tangency of the opposite sides PQ and RS with circle Γ be X and X', let the points of tangency of the opposite sides QR and SP be Y and Y', and let the point of intersection of the tangency chords XX' and YY' be O. If we then apply the theorem of the sum of the angles of a quadrilateral

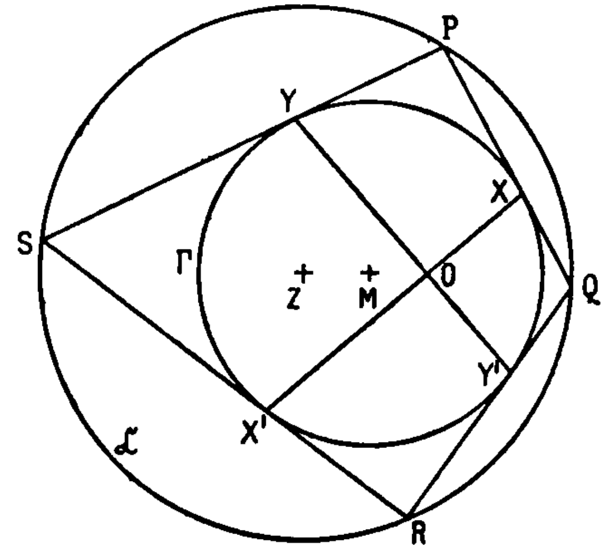

FIG. 31.

to the two quadrilaterals $OXPY$ and $OX'RY'$, designating the quadrilateral angles by means of a line over the letter representing the corner, we obtain the two equations

$$\bar{O} + \bar{X} + \bar{P} + \bar{Y} = 360°, \qquad \bar{O} + \bar{X}' + \bar{R} + \bar{Y}' = 360°.$$

Since the angles $\bar{X}$ and $\bar{X}'$ ($\bar{Y}$ and $\bar{Y}'$) situated at opposite sides of the chord XX' (YY') add up to 180°, addition of the two equations gives the following relation

$$(1) \qquad 2\bar{O} + \bar{P} + \bar{R} = 360°.$$

Now the sum of the two opposite angles $\bar{P}$ and $\bar{R}$ of the chord quadrilateral $PQRS$ is 180°; consequently, $\bar{O} = 90°$.

The tangency chords of the two pairs of opposite sides of a bicentric quadrilateral are therefore perpendicular to each other.

This condition is also sufficient: *A bicentric quadrilateral* PQRS *is obtained if the tangents* PQ, RS, SP, QR *are drawn through the end points* X, X′, Y, Y′ *of two perpendicular chords* XX′ *and* YY′ *of an arbitrary circle* Γ. In fact, it now follows from (1), since $\bar{O} = 90°$, that the sum of the opposite angles $\bar{P}$ and $\bar{R}$ is 180°, i.e., that $PQRS$ is also a *chord* quadrilateral.

The simplest way of obtaining the desired relation between the radii and the axis of the centers of the circumscribed and inscribed circles is by means of the following locus problem. *A right angle is rotated about its fixed vertex, which is located inside a circle; find the locus of* *the point of intersection of the two circle tangents that pass through the point of intersection of the legs of the angle with the circle.*

SOLUTION OF THE LOCUS PROBLEM. Let the given circle be known as Γ, its midpoint as M, its radius as ρ, the fixed vertex of the right angle as O, the distance of the vertex from M as e. Let the legs of the right angle intersect the circle at the (moving) points X and Y; and let the point of intersection of the two circle tangents passing through X and Y be known as P and its distance from the center of the circle as p.

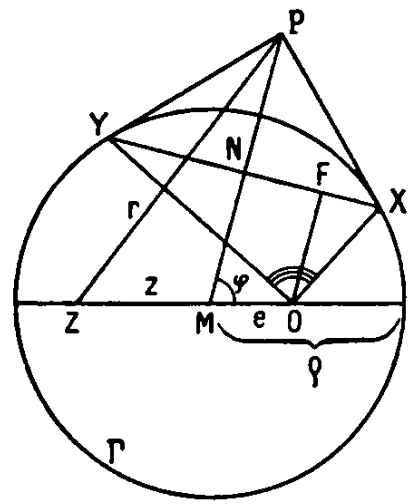

FIG. 32.

We will first determine the relation between p and its angle φ ($= \measuredangle OMP$) with the fixed line MO.

Since OXY is a right triangle,

$$OF^2 = FX \cdot FY,$$

where F represents the base point of the altitude to the hypotenuse. If we introduce the projections $\rho' = MN$ and $e' = e \cos \varphi$ and $\rho'' = NX$ and $e'' = e \sin \varphi$ ($= NF$) on the lines MP and XY, respectively, the equation can be written

$$(\rho' - e')^2 = (\rho'' - e'')(\rho'' + e'')$$

or

$$2\rho'^2 - 2\rho'e' + e'^2 + e''^2 = \rho'^2 + \rho''^2$$

or

$$(2) \qquad 2\rho'^2 - 2\rho'e \cos \varphi + e^2 = \rho^2.$$

Since MXP is a right triangle,

$$MX^2 = MP \cdot MN$$

or

$$(3) \qquad \rho^2 = p\rho'.$$

If we introduce the value of ρ' from (3) into (2), we obtain the relation we are looking for:

$$p^2 + 2\frac{\rho^2 e}{\rho^2 - e^2}p\cos\varphi = \frac{2\rho^4}{\rho^2 - e^2}. \tag{4}$$

The distance $r = ZP$ of a point Z from P on the extension of OM at a distance of $MZ = z$ from M is obtained by the cosine theorem

$$r^2 = z^2 + p^2 + 2zp\cos\varphi. \tag{5}$$

If for z, which up to this point has been arbitrary, we now choose the value

$$MZ = z = \frac{\rho^2}{\rho^2 - e^2}\cdot e, \tag{I}$$

we obtain, in accordance with (4),

$$r^2 = z^2 + \frac{2\rho^4}{\rho^2 - e^2}, \tag{II}$$

and consequently r has a *constant* value!

The desired locus of the point of intersection P *is thus a circle* $\mathfrak{C}$ whose center Z, which is situated on the extension of OM, is determined by (I) and whose radius r is determined by (II).

Naturally, also belonging to this locus are the points of intersection Q, R, S of the tangents, which are obtained when we draw the tangents through the points of intersection of the circle Γ with the extensions of XO and YO.

The quadrilateral $PQRS$ is simultaneously a tangent and chord quadrilateral, in that it circumscribes circle Γ and is inscribed in circle $\mathfrak{C}$. If the right angle XOY is rotated about O so that the points X, Y describe the circle Γ, the quadrilateral $PQRS$ continuously assumes different positions but always circumscribes circle Γ and is always inscribed in circle $\mathfrak{C}$. Similarly, we see that in this way *all* the bicentric quadrilaterals belonging to the two circles Γ and $\mathfrak{C}$ are obtained. The obtained formulas (I) and (II) contain the solution to the problem posed.

We substitute the value obtained from (II) for $\rho^2 - e^2$ in (I) and obtain $e = 2z\rho^2/(r^2 - z^2)$. From this there follows $\rho^2 - e^2 = \rho^2[(r^2 - z^2)^2 - 4\rho^2 z^2]/(r^2 - z^2)^2$. When this value is introduced into (II) we finally obtain the sought-for *relation between the radii* r *and* ρ *and the axis* z *connecting the centers of the circumscribed and inscribed circles of the bicentric quadrilateral:*

$$2\rho^2(r^2 + z^2) = (r^2 - z^2)^2.$$

The developed formula comes from Nicolaus Fuss (1755–1826), a student and friend of Leonhard Euler. Fuss also found the corresponding formulas for the bicentric pentagon, hexagon, heptagon, and octagon (*Nova Acta Petropol.*, XIII, 1798).

The corresponding formula for the triangle had already been given by Euler. It is

$$r^2 - z^2 = 2r\rho$$

and is easily obtained in the following manner. Let ABC be any triangle, let Z and M be the respective centers, r and ρ the radii of the circles of circumscription and inscription, respectively; thus, $ZM = z$ is the axis connecting the centers; further, let D be the point at which the extension of CM meets the circumscribed circle, so that $DM = DA = DB$. The power of the circumscribed circle at M is

$$MC \cdot MD = r^2 - z^2.$$

However, since we can replace sin $(\gamma/2)$ by the ratio ρ/MC as well as by $AD/2r$ or $MD/2r$, $\rho/MC = MD/2r$, i.e.,

$$MC \cdot MD = 2r\rho.$$

When the two values found for the product $MC \cdot MD$ are set equal to each other we obtain Euler's formula.

Note. Much more remarkable than the Fuss formula is a theorem concerning bicentric quadrilaterals that follows directly from the preceding locus consideration. For convenience in expression we will make a prefatory observation.

Let a circle Γ lie completely inside another circle $\mathfrak{C}$. If from any point on $\mathfrak{C}$ we draw a tangent to Γ, extend the tangent line so that it intersects $\mathfrak{C}$, and draw from the point of intersection a new tangent to Γ, extend this tangent similarly to intersect $\mathfrak{C}$, and continue in this manner, we obtain a so-called *Poncelet traverse* which, when it consists of n chords of the larger circle, is called n-sided.

The theorem concerning bicentric quadrilaterals now reads:

If on the circle of circumscription there is one point of origin *for which a four-sided Poncelet traverse is closed, then the four-sided traverse will also close for* any other point of origin *on the circle.*

The French mathematician Poncelet (1788–1867) demonstrated that this theorem is not limited to four-sided traverses only, but is generally true for n-sided traverses, and not only for circles, but for any type of conic section. The general theorem reads: