



# I have a dream ...

By a crossed quadrilateral

In ancient mists of forgotten time  
it has universally been proclaimed  
that all mathematical objects  
have been created equal ...

Nonetheless, because of our shape,  
sides crossing not only at our vertices,  
and our diagonals falling outside,  
we are treated like aliens with no rights  
in the land of our mathematical birth ...

Even though we obey Varignon,  
and despite the sum of our opposite angles  
also being constant when we are cyclic ...  
we are shunned, oppressed and  
discriminated against in apartheid geometry  
because the sum of our interior angles  
is 720, and not 360 degrees ...

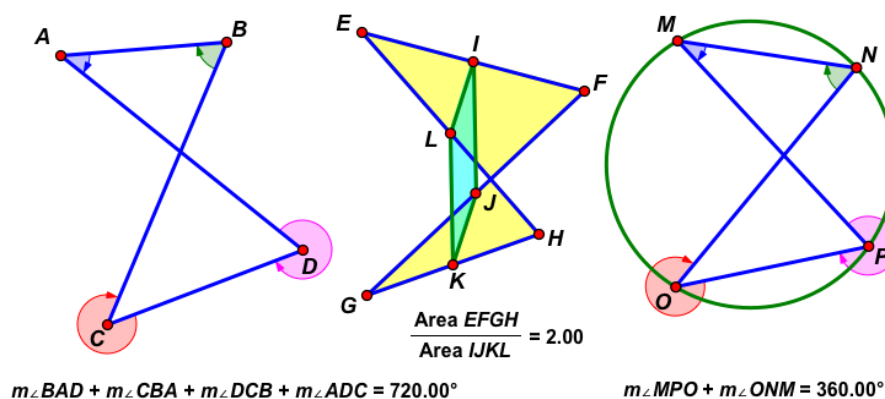
Negative numbers have gained full rights  
and so have imaginary numbers ...  
Today even infinitesimals are citizens,  
as are transcendentals and quaternions  
but we are still considered outcasts ...

Though the dark days of Euclid  
where squares were excluded from  
schools for rectangles and rhombi  
have after 2000 years mercifully passed,  
we crossed polygons are still confined  
to township and ghetto schools.

But I have a dream ... that one day,  
perhaps in a 100 or a 1000 years,  
my children will be finally be free ...  
and live in a nation, where they  
won't be judged by their shape,  
but by their mathematical character ...

I have a dream ...  
that one day, every crossed quadrilateral  
and every crossed polygon  
shall be acknowledged as full citizens  
and no longer be victims of  
prejudiced Lakatosian monster-barring ...

LET FREEDOM REIGN!!!



This bit of prose is intended as a critical satire of the still dominant practice in mathematics education to exclude crossed quadrilaterals from the set of quadrilaterals, and hence from the curriculum.

## CROSSED QUADRILATERALS: A MISSED LAKATOSIAN OPPORTUNITY?

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"... definitions are frequently proposed and argued about when counterexamples emerge ..." – Imre Lakatos (1973:16)

Although mathematics is not an empirical science, it grows and develops, according to Lakatos (1983), similarly to the natural sciences; that is, as a consequence of the quasi-empirical testing of theorems, concepts, definitions, and so forth. New counter-examples necessitate the re-examination of old proofs and definitions, and new proofs and definitions are created accordingly. This process may take place over a long period of time, and a good example is the definition of the concept of ‘function’ which started evolving from about the 17th century into the present day set-theoretic one, even though the function concept was already implicit in much of the work done centuries before by the ancient Greek, Babylonians, Hindus, Chinese, etc.

However, in practice very few teachers or textbooks seem to pay any attention to these ideas or try to incorporate learning activities focused on illustrating a Lakatosian process of the evolution and refinement over time of a definition and/or a proof. Despite the availability of dynamic geometry for over 25 years now, where several counter-examples to traditional definitions and proofs often naturally arise, such counter-examples are persistently ignored by teachers, textbook authors and mathematics education researchers.

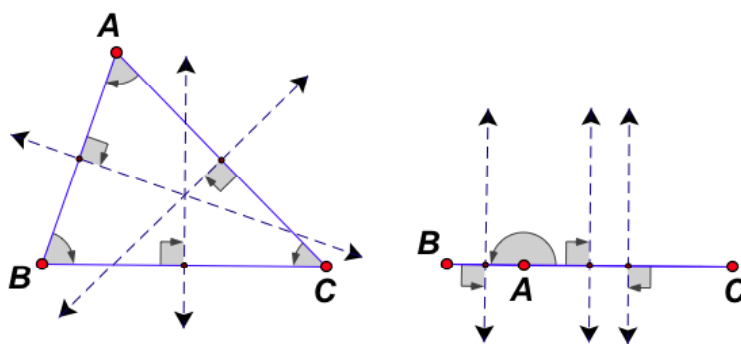


Figure 1

For example, typically a triangle is defined as a plane figure consisting of three non-collinear points connected by three straight-line segments. However, in dynamic geometry a constructed triangle can easily be dragged into a straight line as shown in Figure 1, immediately presenting a natural counter-example to the traditionally given definition of a triangle. The theorem about the  $180^\circ$  triangle angle sum can easily be ‘saved’ by regarding the angle at vertex  $A$  as equal to  $180^\circ$ , and the other two angles as zero. However, to

‘save’ the one about the concurrency of the perpendicular bisectors of its sides will require importing the new definition that ‘parallel lines meet at infinity’, thus showing students that definitions don’t exist immutable out there somewhere, but that they are *created* by mathematicians and are often chosen for ‘convenience’. Surely, a classroom discussion of such a situation can do much to help to dispel the myth that mathematics is an ‘absolutist’ science (Ernest, 1991)?

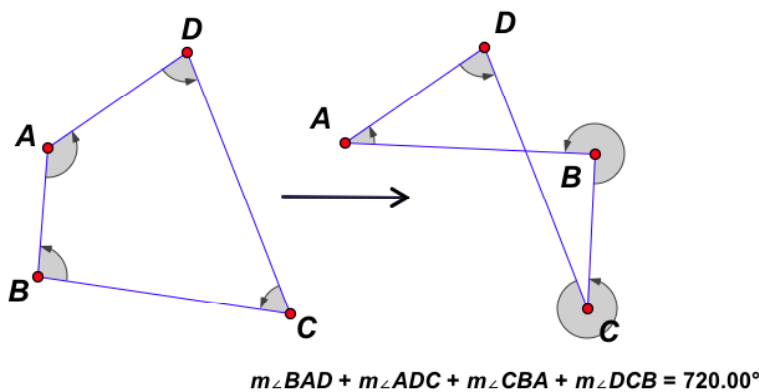


Figure 2

Another example that naturally arises in a dynamic geometry context that can fruitfully be used in the classroom to engage students in reflecting critically upon, and modifying definitions, is that of ‘crossed’ (self-intersecting or bow-tie) quadrilaterals. Consider for example the following dynamic geometry activity from De Villiers (1999) in relation to the well-known theorem that the sum of the angles of a (convex or concave) quadrilateral is always  $360^\circ$ . Construct a quadrilateral  $ABCD$  and measure and sum its (arc) angles. Drag vertex  $D$  over side  $AB$  to obtain a figure similar to the one shown in Figure 2.<sup>1</sup>

This naturally raises the following question for discussion. Why is the sum of the same marked angles no longer equal to  $360^\circ$ , but  $720^\circ$  instead? Is the figure  $ABCD$  a “quadrilateral”? What do we mean by the concept “quadrilateral”? How does this relate to the well-known result formulated above? Why do the two marked angles at  $B$  and  $C$  appear to lie “outside”? What do we mean by “interior” angles? How can we more precisely define “interior” angles of quadrilateral?

As already reported by Galbraith (1979) during individual interviews long before the first dynamic geometry appeared, some students will typically try to ‘monster-bar’, i.e. reject the idea of a crossed quadrilateral as a quadrilateral while some readily accept it. As Lakatos (1983) points out, this is characteristic of refutation by counter-example as it usually depends on the meaning of the terms involved, and consequently definitions are frequently proposed and argued about. Such disagreement in a classroom can therefore lead to a heated discussion under the guidance of a teacher, and as I can attest to many times from personal experience with

<sup>1</sup> An online, dynamic Java sketch with GeoGebra is available at:  
[http://dynamicmathematicslearning.com/crossedquad\\_anglesum.html](http://dynamicmathematicslearning.com/crossedquad_anglesum.html)

both undergraduate as well as post-graduate students, provide students with a valuable, paradigm shifting learning experience.

The issue about what happens to the ‘interior angles’ when a simple closed quadrilateral is dragged into the shape of a crossed quadrilateral is easily resolved with the introduction of ‘directed angles’ (e.g. De Villiers, 1999, p. 26; pp. 115-116). Students may then learn that it is the same sense of maintaining *structure* and *consistency* that guides us to defining  $-1 \times -1 = 1$  when moving from positive integers to negative integers, that also guides us in using directed angles in moving from the interior angles of convex and concave quadrilaterals to those of crossed quadrilaterals.

The proof of the result that the interior angles of a crossed quadrilateral as shown by the second figure in Figure 2 sum to  $720^\circ$  can be done in several different ways (and is left as an exercise to the reader). Students can then simply restate the interior angle sum theorem of quadrilaterals as follows: The interior angle sum of a convex or concave quadrilateral is  $360^\circ$ , but for a crossed quadrilateral it is  $720^\circ$ .

Such a simple, accessible Lakatosian learning experience of modifying, adapting and reformulating a definition and a theorem can help tremendously to combat the pervasive myth that definitions and theorems are always static and fixed. So it is quite sad that generally the potential of engaging with crossed quadrilaterals, and examining their properties, is ignored by textbooks and, hence, by classroom teachers also.

Allowing crossed quadrilaterals as quadrilaterals, of course, also requires the reformulation of some other familiar theorems, but that is also typically part of the process described by Lakatos (1983). For example, for cyclic quadrilaterals one would have to state something like the following: “The sum of the opposite ‘interior’ angles of a convex cyclic quadrilateral is  $180^\circ$ , but for a crossed cyclic quadrilateral it is  $360^\circ$ ” (see Figure 3a).

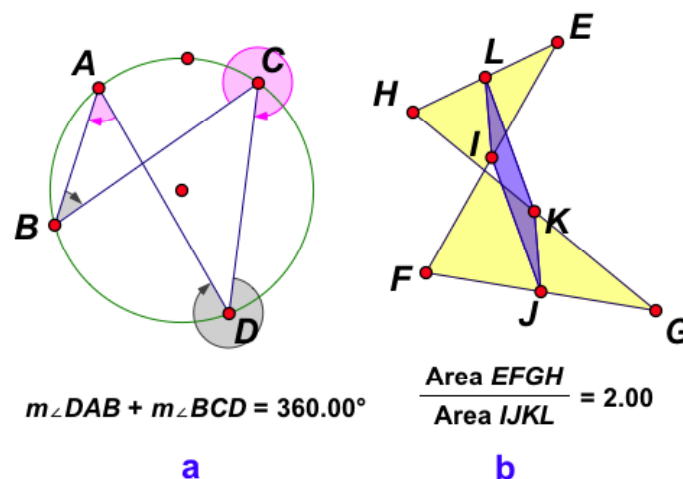


Figure 3

Another instructive example is Varignon's theorem, namely, that the midpoints of any quadrilateral form a parallelogram. This theorem also applies to a crossed quadrilateral, and can easily be proved by students. In addition, the result that the area of the Varignon parallelogram is half that of the original quadrilateral holds even when it becomes crossed in a dynamic geometry context (see Figure 3b). Again this naturally raises the interesting and challenging question about how to define and determine the area of a crossed quadrilateral (e.g. see De Villiers, 2014).

Dealing with crossed quadrilaterals not only provides students with a useful Lakatosian experience, but may also better prepare them for higher geometries such as affine and projective geometry where crossed polygons regularly appear. For example, Pascal's theorem for a hexagon inscribed in a conic that the three points of intersection of the opposite sides are collinear (lie in a straight line) apply equally to a crossed hexagon as shown in Figure 4.

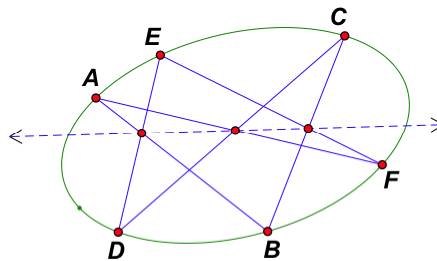


Figure 4

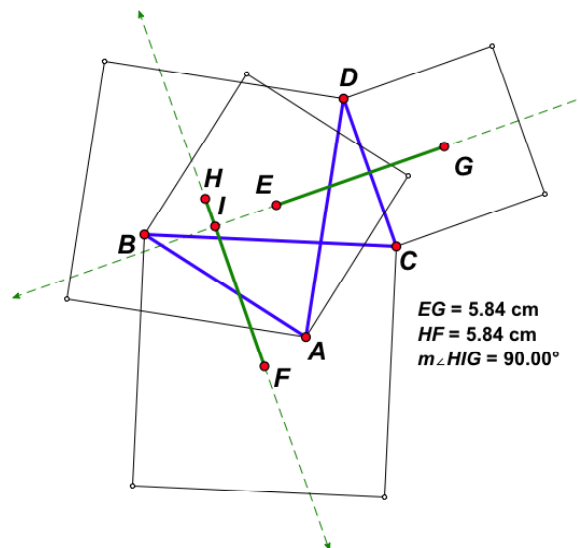


Figure 5

Many other beautiful theorems in relation to quadrilaterals also hold for crossed quadrilaterals. Perhaps most notable is Van Aubel's theorem, that states that the line segments joining the centres of the squares on opposite sides of a quadrilateral are perpendicular to each other and equal in length (see the crossed quadrilateral<sup>2</sup> in Figure 5).

<sup>2</sup> The reader is invited to visit a dynamic online Java sketch of Van Aubel's theorem at: <http://dynamicmathematicslearning.com/aubelparm.html>, dragging it into the shape of a crossed quadrilateral.

Providing students with the Lakatosian experience of engaging with crossed quadrilaterals also exposes them to a situation where their geometric intuition is likely to fail, and how in such a situation one has to carefully rethink and define, for example, the concept of ‘interior angle’, etc. in a consistent way so as to be able to extend it to the unfamiliar, non-intuitive domain. Such an experience therefore nicely illustrates the important general mathematical principle of *maintaining structural consistency of definitions* when moving from a familiar to an unfamiliar, non-intuitive context in mathematics. The same sense of maintaining structure and consistency is applied when moving from operations for positive whole numbers to negative whole numbers, then further towards fractions, and eventually to complex numbers and quaternions, etc.

Though there appears to be fairly widespread consensus among the mathematics education community about the validity and value of a Lakatosian perspective on mathematics, it is strange that very little of this sentiment translates into specifically designed learning experiences that show and engage students with the evolution and modification of definitions and theorems over time. In most cases textbooks and teachers typically introduce students to concepts with ready-made, polished definitions, and no opportunity is created to show either the process of defining or how definitions may change and become more precise in light of new, challenging examples. However, rather than confining crossed quadrilaterals to the proverbial mathematical trashcan, utilizing their strange, non-intuitive properties can be a positive step in incorporating some of the ideas of Lakatos into the classroom.

## REFERENCES

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- De Villiers, M. (1999). *Rethinking Proof with Sketchpad*. Emeryville: Key Curriculum Press.
- De Villiers, M. (2014). Slaying a Geometrical Monster: Finding the Area of a Crossed Quadrilateral. *The Scottish Mathematical Council Journal*, 44, pp. 71-74. (Available to download at: <http://dynamicmathematicslearning.com/crossed-quad-area.pdf>).
- Ernest, P. (1991). *The Philosophy of Mathematics Education*, London, Falmer Press.
- Lakatos, I. (1983). *Proofs and refutations*, 4th reprint. Cambridge: Cambridge University Press.

## FURTHER RESOURCES

On mathematical properties of crossed polygons:

[http://dynamicmathematicslearning.com/crossedquad\\_anglesum.html](http://dynamicmathematicslearning.com/crossedquad_anglesum.html)

Michael de Villiers' homepage: <http://dynamicmathematicslearning.com/homepage4.html>

Dynamic Geometry Sketches: <http://dynamicmathematicslearning.com/JavaGSPLinks.htm>