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Equi-angled cyclic and equilateral circumscribed polygons

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"*Man muss immer generalisieren*." (One should always generalize). – Karl Jacobi (1804-1851). In P. Davis and R. Hersh, 1981, *The Mathematical Experience*, Boston: Birkhauser, p. 134.

Some three to four years ago the author discovered the following two elementary, yet interesting results with the aid of *Sketchpad*. It nicely extends familiar results in high school geometry (for example, for a rectangle and a rhombus), and would be easily accessible for students at high school and undergraduate level. The results can also be proved in a number of different ways.

Theorem 1

A cyclic polygon has all angles equal, if and only if, the alternate sides are equal. (For n odd, it becomes regular).

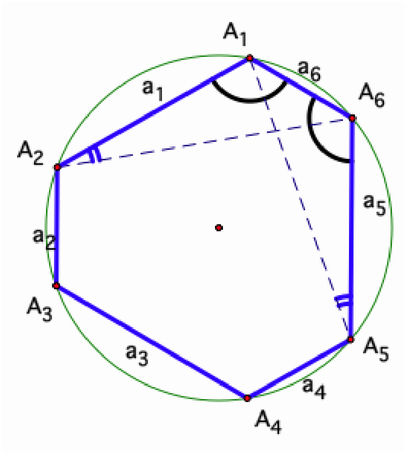

Figure 1

Proof

Consider the equi-angled, convex cyclic hexagon in Figure 1 as a generic example for n even. Triangles $A_1A_2A_6$ and $A_6A_5A_1$ are congruent, since A_1A_6 is common, $\angle A_2A_1A_6 = \angle A_5A_6A_1$ (given) and $\angle A_1A_2A_6 = \angle A_6A_5A_1$ (on same chord A_1A_6). Thus, $a_1 = a_5$.

In exactly the same way can be shown that $a_3 = a_5$ and $a_1 = a_3$, from which it follows that all the odd-numbered sides are equal. Similarly, it follows from triangle congruency that all the even-numbered sides are also equal.

Conversely, it also follows that if a convex cyclic hexagon has all the odd and even-numbered sides equal, then it has all angles equal. For example, in Figure 1 triangles $A_1A_2A_6$ and $A_6A_5A_1$ remain congruent ($\angle$, $\angle$, s), etc.

However, since the same argument above applies when n is odd, it obviously follows by going around cyclically round the cyclic polygon, that all sides will be equal (i.e. the polygon becomes regular). More over, the arguments equally apply to crossed cyclic polygons.

Theorem 2

A circumscribed polygon has all sides equal, if and only if, the alternate angles are equal. (For n odd, it becomes regular).

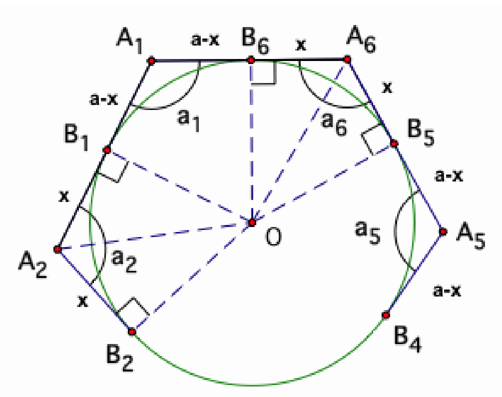

Figure 2

Proof

Consider the partially completed equilateral circumscribed hexagon in Figure 2 as a generic example for n even. Assume that all sides are of length a. Using the result that the tangents from a point outside to the circle are equal, label the different segments of the sides x and $a - x$ as shown. Then triangles A_2B_1O and A_6B_6O are congruent (s, $\angle$, s). This implies that the kites $B_1A_2B_2O$ and $B_6A_6B_5O$ are congruent. Thus, $\angle a_2 = \angle a_6$. Since there is an even number of sides, we can as before continue in the same manner to prove that $\angle a_2 = \angle a_4 = \angle a_6$ and $\angle a_1 = \angle a_3 = \angle a_5$.

Conversely, it follows similarly that if a convex circumscribed hexagon has all its alternate angles equal, then all its sides are equal. For example, one can similarly show that if $\angle a_2 = \angle a_6$, then the kites $B_1A_2B_2O$ and $B_6A_6B_5O$ are congruent, etc.

However, since the same arguments above apply when n is odd, it obviously follows by going around cyclically round the circumscribed polygon, that all angles will be equal (i.e. the polygon becomes regular). More over, the arguments equally apply to crossed circumscribed polygons.

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Side-angle Duality

"Symmetry as wide or as narrow as you may define it, is one idea by which man through the ages has tried to comprehend, and create order, beauty and perfection."

- Hermann Weyl

The above two theorems display an interesting duality between "*sides*" and "*angles*". Though not a generally valid duality in plane geometry, it nonetheless appears quite often, and has been fruitful in some cases to conjecture new results as mentioned in [1].

In both cases above, when n is odd, regular polygons are obtained, and hence are obviously self-dual with respect to sides and angles. We will therefore here look a little more closely at the perhaps more interesting side-angle duality of the polygons with n even.

The rectangle and rhombus are the simplest examples of Theorems 1 and 2 and display the side-angle duality as shown in the table below. Note that the concepts of 'equal' and 'perpendicular' diagonals also appear dual since connecting the midpoints of the sides of a quadrilateral with perpendicular diagonals produces a rectangle whereas connecting the midpoints of the sides of a quadrilateral with equal diagonals produces a rhombus.

Rectangle	**Rhombus**
All *angles* equal	All *sides* equal
Alternate *sides* equal	Alternate *angles* equal
Circumscribed circle (*cyclic*)	Inscribed circle (*circumscribed*)
An axis of symmetry through each pair of opposite *sides*	An axis of symmetry through each pair of opposite *angles (vertices)*
Diagonals are equal in *length*	Diagonals intersect at equal *angles*

For n even, some examples of equi-angled cyclic polygons are shown in Figure 3, starting with the special case of a rectangle, and in Figure 4, some examples of equilateral circumscribed polygons are shown, starting with the special case of a rhombus. Note that in both figures, k represents the *total turning* one would undergo walking around the perimeter of each figure, and is measured by the number of revolutions undergone.

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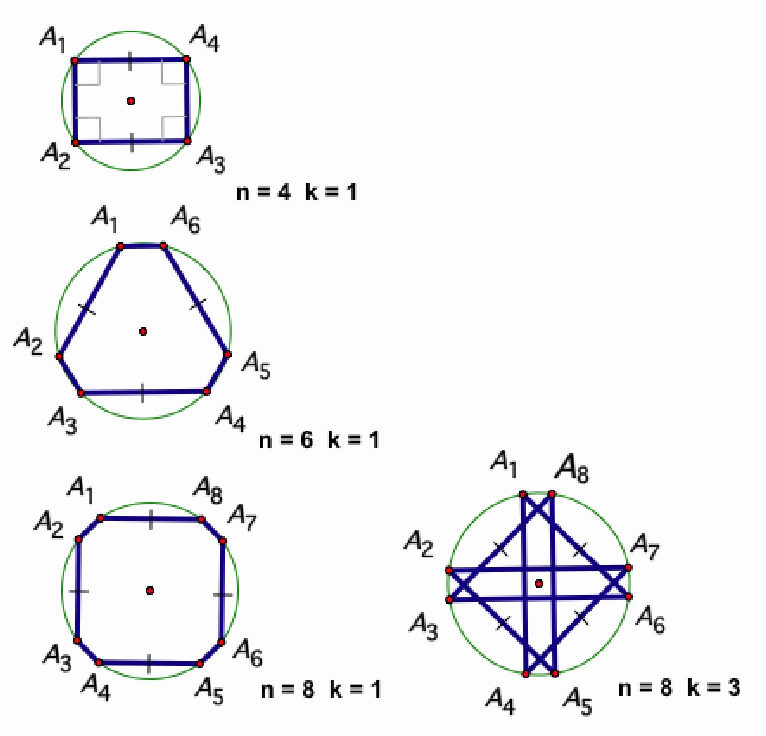

Figure 3

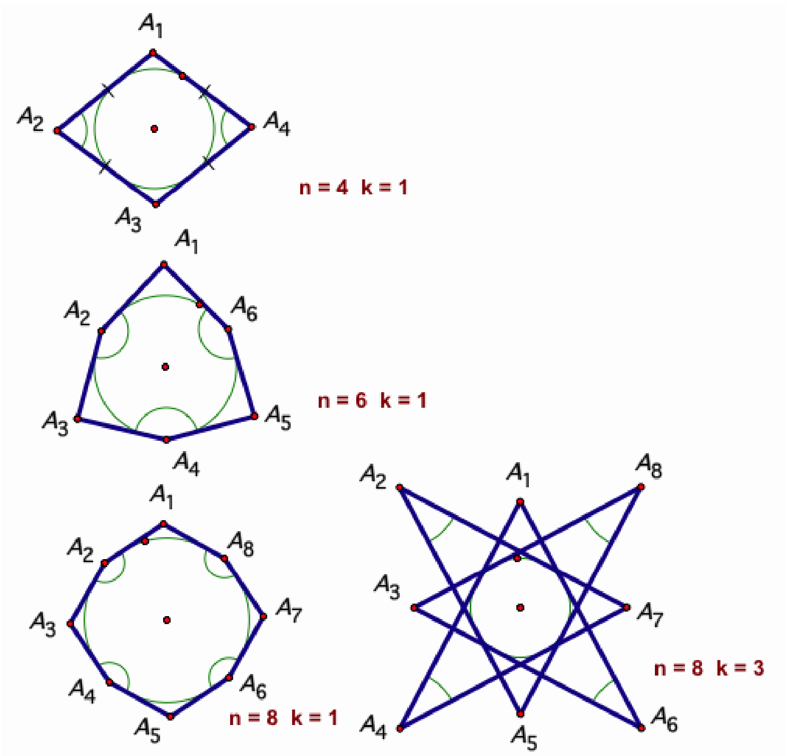

Figure 4

Visually, or from accurate construction and measurement with dynamic geometry software, it is easy to verify that all the side-angle properties for a rectangle and rhombus, in general also hold for equi-angled cyclic and equilateral circumscribed polygons for n even. The

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duality is further maintained by the additional property given in the table below.

Equi-angled cyclic for *n* even	**Equilateral circumscribed for *n* even**
Opposite *sides* equal if *n*/2 is even	Opposite *angles* equal if *n*/2 is even

Defining and Systematizing

In order to define mathematical objects, it is standard practice to select necessary and sufficient properties from the total set of properties of the object, and usually our eventual choice of definition is a matter of convenience. One way of defining equi-angled cyclic and equilateral circumscribed polygons for *n* even could of course be by means of the results contained in Theorems 1 and 2. But that provides relatively cumbersome formulations, and proving that all the other properties mentioned in the tables above, logically follow from such definitions, though elementary, is not entirely straightforward.

As is the case for rectangles and rhombi, it seems the most convenient, elegant definitions for their respective dual generalizations to – let's agree to call them, respectively *semi-regular angle*-gons and *semi-regular side*-gons – would be in terms of symmetry, respectively as follows:

Semi-regular angle-gon	**Semi-regular side-gon**
"A semi-regular *angle*-gon is any polygon with *n* even and an axis of symmetry through each pair of opposite *sides*."	"A semi-regular *side*-gon is any polygon with *n* even and an axis of symmetry through each pair of opposite *angles* (vertices)."

From these two definitions, it now almost immediately follows from symmetry that a semi-regular angle polygon is cyclic (i.e. the concurrency of the axes of symmetry shows there is a point, equi-distant from all the vertices), and that by repeated reflections it has all angles, alternate sides, diagonals equal, etc., and likewise for a semi-regular side polygon.

Notes

1. A dynamic online version of Figures 3 and 4 are available at: http://dynamicmathematicslearning.com/semi-regular-anglegon.html
2. A dynamic geometry (*Sketchpad 4*) sketch in zipped format (Winzip) of an equi-angled cyclic hexagon and equilateral, circumscribed hexagon, as well as a respective classification of different kinds of equi-angled cyclic and equilateral

circumscribed polygons for *n* even, can be downloaded directly from:

http://mysite.mweb.co.za/residents/profmd/equi-anglecyclichex.zip

(If not in possession of a copy of *Sketchpad 4/5*, these sketches can be viewed/interacted with a free demo version of *Sketchpad 5* that can be downloaded from: http://www.keypress.com/x24795.xml)

3. Usually an *n*-gon with a total turning of *k* is economically denoted by the Schäfli notation $\{n/k\}$.

References

1. M. de Villiers, Dual generalizations of Van Aubel's theorem, *Mathematical Gazette*, **82**(496), (Nov. 1998) pp. 405-412.

(Available from http://mzone.mweb.co.za/residents/profmd/aubel2.pdf)

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NOTE: Also read about further generalizations of these polygons to "alternate sides cyclic-2*n*-gons and "alternate angles circum-2*n*-gons", which are respective generalizations of isosceles trapezia and kites in my July 2011 **Feedback Note** at:

http://dynamicmathematicslearning.com/equi-angle-equi-side-general.pdf