CHAPTER SIX

Geometrical Fallacies

THE fallacies of geometry are more remarkable than those of algebra in at least one respect, for the deception is not only of the mind, but of the eye as well. The diagrams in Figure 18 at the beginning of Chapter 4 showed how easy it is for the eye to mislead the mind. The examples in the last chapter furnished scant material for the eye, but did reveal that care must be used if the mind is not to lead itself astray. Formal deduction in geometry is to some extent a combination of seeing and reasoning, for in the proof of any theorem the logical processes of the mind are guided by and checked against what the eye sees in the figure.

It may be of interest to note that Euclid compiled a collection of exercises for the detection of fallacies, but unfortunately this work has been lost.[1]

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As our first example of one type of fallacious geometrical reasoning we shall carry through a complete discussion of this remarkable theorem:

To prove that any triangle is isosceles.[2]

Let ABC be any triangle, as in Figure 62(a). Construct the bisector of $\angle C$ and the perpendicular bisector of side AB. From G, their point of intersection, drop perpendiculars GD and GF to

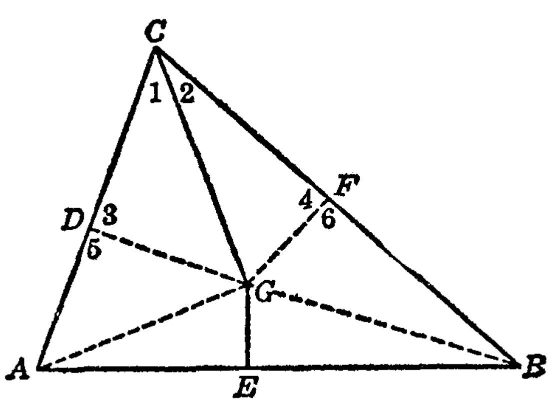

FIG. 62(a)

AC and *BC* respectively and draw *AG* and *BG*. Now in triangles *CGD* and *CGF*, $\angle 1 = \angle 2$ by construction and $\angle 3 = \angle 4$ since all right angles are equal. Furthermore the side *CG* is common to the two triangles. Therefore triangles *CGD* and *CGF* are congruent – can be made to coincide, that is. (If two angles and a side of one triangle are equal respectively to two angles and a side of another, the triangles are congruent.) It follows that $DG = GF$. (Corresponding parts of congruent triangles are equal.) Then in triangles *GDA* and *GFB*, $\angle 5$ and $\angle 6$ are right angles and, since *G* lies on the perpendicular bisector of *AB*, $AG = GB$. (Any point on the perpendicular bisector of a line is equidistant from the ends of the line.) Therefore triangles *GDA* and *GFB* are congruent. (If the hypotenuse and another side of one right triangle are equal respectively to the hypotenuse and another side of a second, the triangles are congruent.) From these two sets of congruent triangles – *CGD* and *CGF*, and *GDA* and *GFB* – we have, respectively,

$$CD = CF \tag{1}$$

and

$$DA = FB. \tag{2}$$

Adding (1) and (2), we conclude that $CA = CB$, so that triangle *ABC* is isosceles by definition.

It may be argued that we do not know that *EG* and *CG* meet within the triangle. Very well, then, we shall examine all other possibilities. The above proof, word for word, is valid in the cases wherein *G* coincides with *E*, or *G* is outside the triangle but so near to *AB* that *D* and *F* fall on *CA* and *CB* and not on *CA* and *CB* produced. These cases are illustrated in Figures 62(b) and (c).

There remains the possibility, shown in Figure 62(d), in which

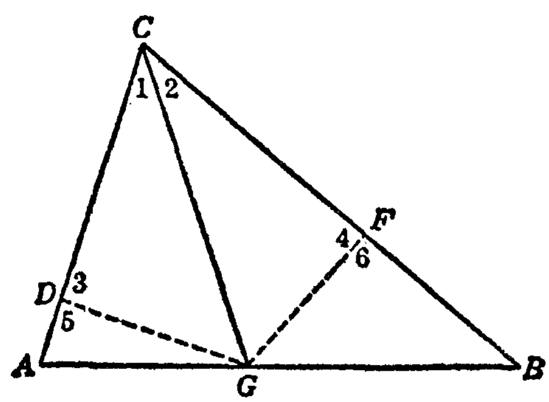

FIG. 62(b)

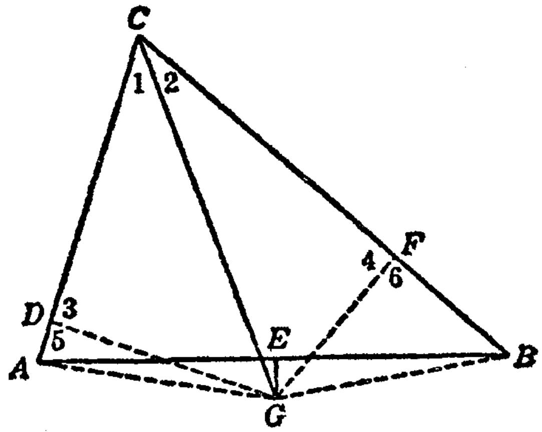

FIG. 62(c)

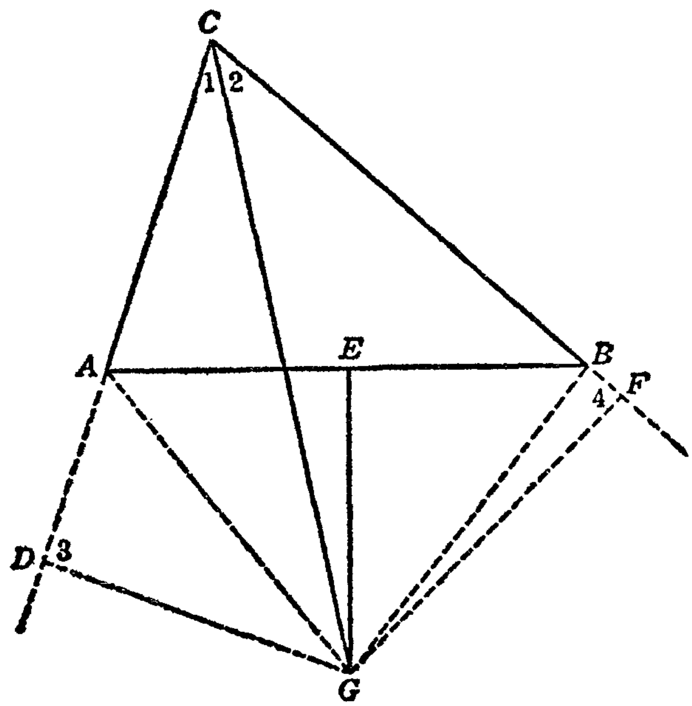

FIG. 62(d)

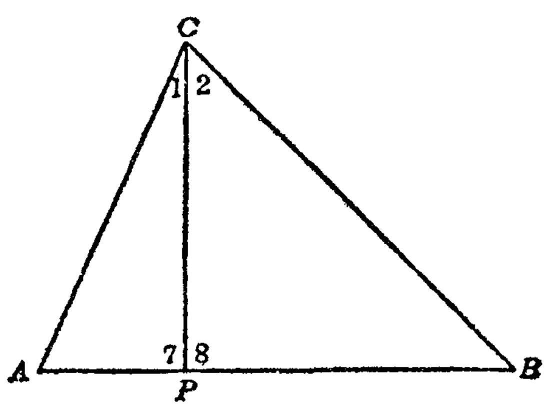

FIG. 62(e)

G lies so far outside the triangle that *D* and *F* fall on *CA* and *CB* produced. Again, as in the first case, triangles *CGD* and *CGF* are congruent, as are triangles *GDA* and *GFB*. And again $CD = CF$ and $DA = FB$. But in the present case we must subtract these last two equations in order to have $CA = CB$.

Finally, it may be suggested that *CG* and *EG* do not meet in a single point *G*, but either coincide or are parallel. A glance at Figure 62(e) shows that in either of these cases the bisector *CP* of angle *C* will be perpendicular to *AB*, so that $\angle 7 = \angle 8$. Then $\angle 1 = \angle 2$, *CP* is common, and triangle *APC* is congruent to triangle *BPC*. Again $CA = CB$.

It certainly *appears* that we have exhausted all possibilities and that we must accept the obviously absurd conclusion that all triangles are isosceles. There is one more case, however, which may be worth investigating. Is it not possible for *one* of the points *D* and *F* to fall *inside* the triangle and for the *other* to fall *outside*? A correctly drawn figure will indicate that this possibility is indeed the only one. Furthermore we can prove it as follows.

Circumscribe a circle about the triangle *ABC*, as in Figure 62(f). Since $\angle 1 = \angle 2$, *CG* must bisect arc *AB*. ($\angle 1$ and $\angle 2$ are inscribed

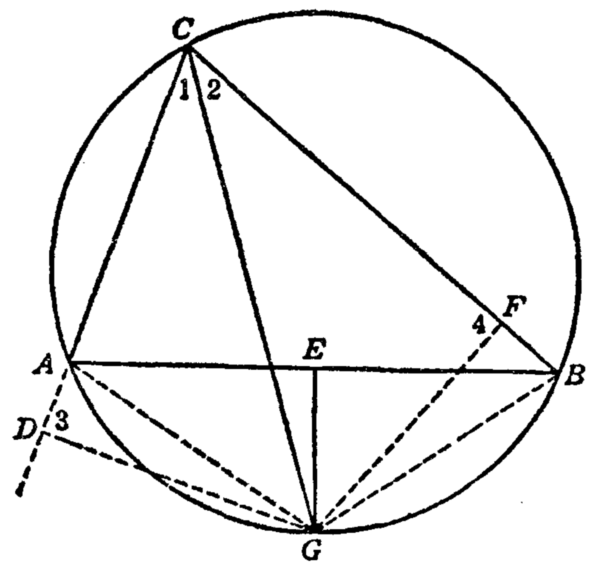

FIG. 62(f)

angles and, being equal, must be subtended by equal arcs.) But *EG* also bisects arc *AB*. (The perpendicular bisector of a chord bisects the arc of the chord.) It follows that *G* lies on the circumscribed circle and that *CAGB* is an inscribed quadrilateral. Now

$\angle CAG + \angle CBG$ is a straight angle. (The opposite angles of an inscribed quadrilateral are supplementary.) But if $\angle CAG$ and $\angle CBG$ were both right angles, D and F would coincide with A and B respectively; so the conclusion that $CD = CF$ (a conclusion established in the first case) would reduce to $CA = CB$, which is contrary to our hypothesis that ABC is *any* triangle. Consequently one of the angles CAG and CBG must be acute and the other obtuse, which means that either D or F (D in the figure) must fall outside the triangle and the other inside. The relations $CD = CF$ and $DA = FB$ are true here, as they were in all of our other cases. But whereas $CB = CF + FB$, we now have $CA = CD - DA$, not $CD + DA$.

This discussion has been lengthy, but it should have been instructive. It shows how easily a logical argument can be swayed by what the eye sees in the figure and so emphasizes the importance of drawing a figure correctly, noting with care the relative positions of points essential to the proof. Had we at the start actually constructed – by means of ruler and compasses – the angle bisector and the various perpendiculars, we should have saved ourselves a good deal of trouble.

The following five fallacies are all concerned with the same pitfalls as the one we have just worked over in detail. Watch your step!

Paradox 1. To prove that there are two perpendiculars from a point to a line.[3]

Let any two circles intersect in Q and R. Draw diameters QP and QS and let PS cut the circles at M and N respectively, as in Figure 63. Then $\angle PNQ$ and $\angle SMQ$ are right angles. (An angle

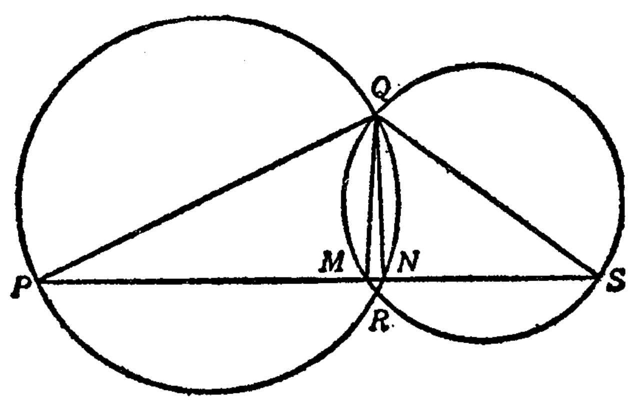

FIG. 63