Pair Eight: With magic sum 110

11	24	32	43
33	42	14	21
44	31	23	12
22	13	41	34

11	42	23	34
33	24	41	12
44	13	32	21
22	31	14	43

The above procedure works for any order except 2, 3, and 6. It has been shown that orthogonal diagonal Latin squares exist for every finite order except 2, 3 and 6 (for more details, see [6, 7]).

Acknowledgment: I would like to thank the referees for their suggestions and comments.

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91.50 A generalisation of Feynman's triangle

Introduction

The original note on Feynman's triangle [1] stimulated considerable interest and generated two responses [2, 3]. These introduced the general Feynman triangle UVW for ΔABC as shown in Figure 1 in which

$$\frac{BL}{LC} = \frac{1}{l}, \qquad \frac{CM}{MA} = \frac{1}{m}, \qquad \frac{AN}{NB} = \frac{1}{n} \tag{1}$$

and H, h_U, h_V, h_W are the vertical heights of A, U, V, W above BC respectively. This note is concerned mainly with the case $l = m = n = p$.

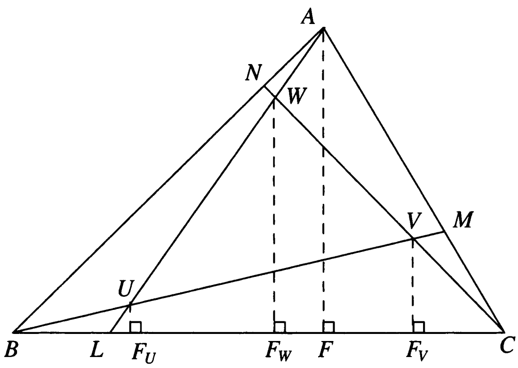

FIGURE 1

The following formulae are required (with the standard notation of ΔABC):

$$\frac{\Delta UVW}{\Delta} = \frac{(p-1)^2}{p^2+p+1} = S(p) \qquad (2)$$

(given in the Feedback, Feynman's triangle corresponding to the case $p = 2$).

$$\frac{h_U}{H} = \frac{1}{p^2+p+1}, \qquad \frac{h_V}{H} = \frac{p}{p^2+p+1}, \qquad \frac{h_W}{H} = \frac{p^2}{p^2+p+1} \qquad (3)$$

$$\Delta UAB = \Delta VBC = \Delta WCA = \frac{p\Delta}{p^2+p+1} \qquad (4)$$

$$\left(p^2+p+1\right)^2 WU^2 = (p-1)^2\left[(pc+b\cos A)^2 + b^2\sin^2 A\right] \qquad (5)$$

$$\left(p^2+p+1\right)\cot\angle WUV = (p+1)\cot A + p(p+1)\cot B - p\cot C \qquad (6)$$

For more information see the postscript.

The distinctive ranges of p

The graph of $S(p)$ is continuous as the equation $p^2 + p + 1 = 0$ has no real roots and the turning points are given via

$$\frac{dS}{dp} = \frac{3(p-1)(p+1)}{(p^2+p+1)^2} = 0,$$

where the minimum $S(1) = 0$ corresponds to $S(p) \geqslant 0$ and $S(-1) = 4$ is the maximum. The curve is shown in Figure 2, which identifies the ranges $p \geqslant 1$, $1 > p \geqslant 0$, $0 > p > -1$, $-1 \geqslant p$, with $BL = BC$ when $p = 0$.

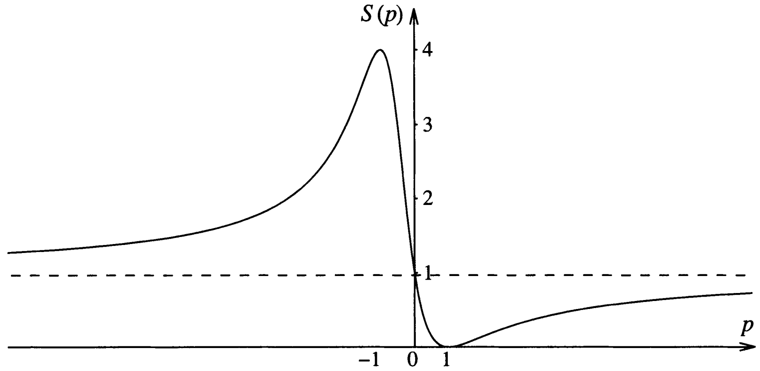

FIGURE 2

The inverse formula for p

The equation $S(p) = m$, which is valid for $0 < m < 4$, with $m \neq 1$, has either two positive roots or two negative roots given by

$$p^2 - 2kp + 1 = 0 \qquad \text{with} \qquad k = \frac{2 + m}{2(1 - m)}. \tag{7}$$

As the product of the roots is one, $S\left(\frac{1}{p}\right) = S(p)$, which is easily checked in (2). The inverse formula is $S\left(k \pm \sqrt{k^2 - 1}\right) = m$ with $k - \sqrt{k^2 - 1} = 1/\left(k + \sqrt{k^2 - 1}\right)$.

For example

$$S(2) = \frac{1}{7} = S\left(\frac{1}{2}\right) \text{ and } S\left(-\frac{1}{2}\right) = 3 = S(-2).$$

The positive ranges

Figure 1 has been drawn to illustrate the range $p \geqslant 1$. Notice that the diagram clearly shows

$$h_U < h_V < h_W < H$$

in agreement with (3). When $p = 1$, L, M and N are the mid-points of their respective sides and ΔUVW reduces to the centroid G, which is consistent with (2) and (5). From (3), in this case

$$h_U = h_V = h_W = \frac{H}{3} = h_G,$$

which is a special case of the general result

$$h_U + h_V + h_W = H.$$

From (6), $\cot\angle AGB' + \cot\angle BGC' + \cot\angle CGA' = \cot A + \cot B + \cot C$, where A', B' and C' are the midpoints of BC, CA and AB. This identity can be checked independently and is a useful exercise for the cotangent rule [4].

The configuration for the range $1 > p \geqslant 0$ is shown in Figure 3 where $BL > LC$ etc. and the notation is consistent with Figure 1.

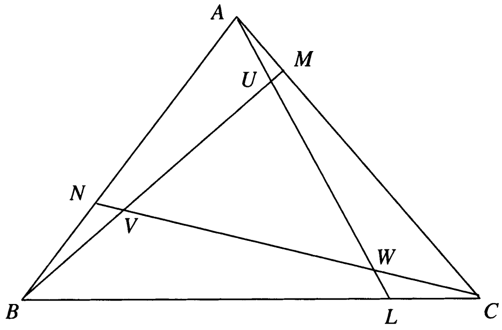

FIGURE 3

In this case, $h_W < h_V < h_U < H$, which is again consistent with (3). When $p = 0$, $LC = MA = NB = 0$ and ΔUVW is coincident with ΔABC, which is consistent with (3), (5) and (6). Following on from (7) any ΔUVW in Figure 3 has the same area as one of the triangles UVW in Figure 1. Identify these triangles with a suffix corresponding to their Figure so that

$$\frac{BL_1}{L_1C} = \frac{1}{p_1} \quad \text{and} \quad \frac{BL_3}{L_3C} = \frac{1}{p_3} = p_1 \quad \text{hence} \quad \frac{CL_3}{L_3B} = \frac{1}{p_1}.$$

So, not surprisingly, a Feynman triangle can be defined by specifying the ratios in a clockwise sense from C rather than anticlockwise from B. This is implied in [1]. In general, these two triangles are not congruent, which only occurs when ΔABC is isosceles as can be verified by (5) and (6). However, when the two triangles are drawn in the same ΔABC then $V_1V_3 /\!/ BC /\!/ U_1W_3 /\!/ W_1U_3$, which can be confirmed by the height formula (3).

The negative ranges

In the range $0 > p > -1$, the negativity of p shows that L, M and N are external to their sides. As $BL > CL$ so L is on the extension beyond C and similarly for M and N as shown in Figure 4.

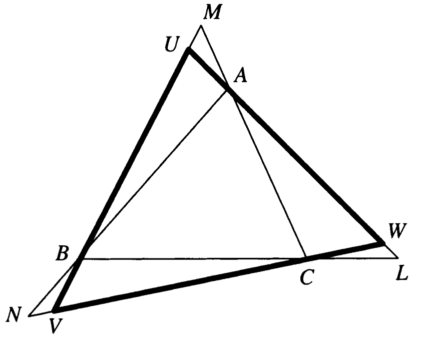

FIGURE 4

As $p \to 0$ so L moves back towards C in comparison with its forward movement in Figure 2. From (4) the external triangles UAB, VBC and WCA are algebraically negative so that

$$\Delta ABC - 3\Delta UAB = \Delta UVW$$

is greater than ΔABC for Figure 4, whereas it is less than ΔABC for Figure 2. The simple result $S\left(-\frac{1}{2}\right) = 3$ corresponds to $h_W = \frac{1}{3}H$ and it is a useful exercise to draw the relevant diagram and to give alternative proofs by similar triangles and the cotangent rule.

The points L, M and N are also external to their sides in the final range $-1 \geqslant p$, but now they lie on the other extensions. A diagram will show ΔUVW tilted in the opposite sense to Figure 3. There is a discontinuity in p on AB with L moving towards B from the left as $p \to -\infty$, but L moving to B from the right in Figure 1 as $p \to +\infty$. Again there is a family of parallel lines when the two triangles of equal area for both negative ranges are drawn on the same ΔABC. Again these can be identified by suffices corresponding to Figures 3 and 4.

The area of ΔUVW is maximum with $S(-1) = 4$ when $\Delta ABC = \Delta = \Delta UAB = \Delta VBC = \Delta WCA$. From $\dfrac{BL}{LC} = \dfrac{1}{p}$, $BL = \dfrac{a}{p+1}$, so L is at an infinite distance from B, which agrees with $LB = LC$ as BC has become insignificant. This implies $AL /\!/ CB$ and similarly $BM /\!/ AC$, $CN /\!/ BA$ as shown in Figure 5. To appreciate this limiting configuration, it is helpful to draw a small triangle ABC so that BL, CM and AN can be made relatively long. The triangle of maximum area is shown in Figure 5.

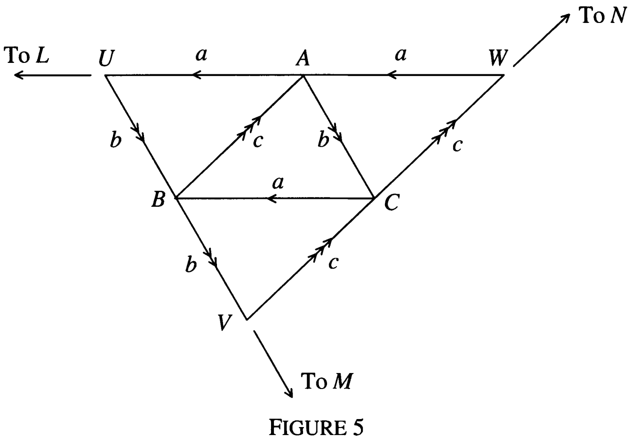

FIGURE 5

This diagram clearly satisfies the area requirements of the triangles and agrees with (3), (5) and (6).

Postscript

The formulae corresponding to (3), (4), (5) and (6) for the general ratios (1) are:

(i) $\dfrac{h_U}{H} = \dfrac{1}{ml + m + 1}$ and $\dfrac{h_W}{H} = \dfrac{ln}{ln + l + 1}$

from a combination of Menelaus' theorem for AU / UL and AW / WL with similar triangles.

(ii) $\Delta WCA = \Delta ALC - \Delta WLC = \dfrac{l\Delta}{ln + l + 1}$

and, from cyclic change to ΔVBC, $\dfrac{h_V}{H} = \dfrac{n}{nm + n + 1}$.

(iii) $$\Delta UVW = \Delta ABC - (\Delta UAB + \Delta VBC + \Delta WCA) = \frac{(lmn - 1)^2 \Delta}{(ln + l + 1)(ml + m + 1)(nm + n + 1)},$$

which is called Routh's theorem in [3].

(iv) $$\frac{(ln + l + 1)^2 WU^2}{(lmn - 1)^2} = \frac{(lc + b\cos A)^2 + b^2 \sin^2 A}{(ml + m + 1)^2}$$

from $WU = AU - AW$ in terms of AL, with AL^2 given by generalising Apollonius' theorem for ΔABC.

(v) $(ml + m + 1)\cot\angle WUV = (l + 1)\cot A + l(m + 1)\cot B - m\cot C$

from the application of the cotangent rule to ΔABL and ΔABC.

Acknowledgement

This note has been rewritten and considerably improved following the referee's suggestion that the original should be reorganised.

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91.51 Heron triangles with $\angle B = 2\angle A$

This note characterises all Heron triangles *ABC* (that is, triangles *ABC* with integer sides and integer area) having $\angle B = 2\angle A$. Deshpande [1] showed that triangles with sides

$$a = n^2,\ b = nm,\ c = m^2 - n^2, \qquad (1)$$

for integer m, n with $0 < n < m < 2n$, have $\angle B = 2\angle A$ (denoting sides