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On Feynman's Triangle problem and the Routh Theorem

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In this article, we give a brief history of the Feynman's Triangle problem and describe a simple method to solve a general version of this problem, which is called the Routh Theorem. This method could be found useful to school teachers, instructors or lecturers who are involved in teaching geometry.

1. Introduction

According to Cook and Wood (2004), the one-seventh area triangle problem was posed to the physicist Richard Feynman by Prof. Kai-Li Chung of Stanford University in a dinner shortly after a colloquium at Cornell University. It is called the Feynman Triangle problem now.

Feynman's Triangle Problem

For a triangle in the plane, if each vertex is joined to the point one-third along the opposite side (measured say, anticlockwise), prove that the area of the inner triangle formed by these lines is exactly one-seventh of the area of the initial triangle (Fig. 1).

Although Richard Feynman thought the result was wrong and tried to disprove it at the beginning, he finally proved it in the special case when the triangle was equilateral. Since then, this problem became quite famous and could often be found in academic articles, textbooks or mathematics contests from time to time. For instance, Cook and Wood (2004) described several methods to solve this problem, namely, by means of vectors, dissection, the Menelaus Theorem and Affine Geometry. On the other hand, some authors attempted to deal with a more general case when each vertex was joined to the point $1/p$ ($p > 2$) along the opposite side of the triangle (Villiers, 2005; Clarke, 2007). However, this problem is related to an interesting result called the Routh Theorem in plane geometry, which was described in *A Treatise on Analytical Statics with Numerous Examples* (Volume I) by Edward Routh (1909), and later in *Introduction to Geometry* by Harold Coxeter (1969). Some recent works related to this theorem can be found in the article by Man (2007) and the book by Pech (2007). In this article, we describe how to use a simple result in plane geometry to prove the Routh Theorem, which can be accessible to students even at junior secondary level. We hope our findings will be found useful to school teachers, instructors or lecturers who are involved in teaching geometry.

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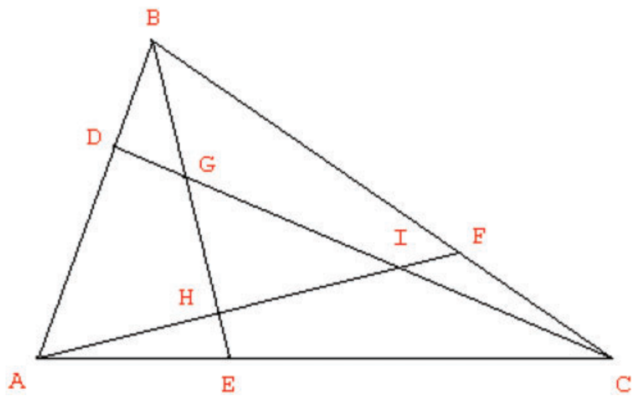

FIG. 1. Feynman's Triangle. (This figure appears in colour in the online version of Teaching Mathematics and its Application.)

2. A simple result in plane geometry

In plane geometry, there is a basic result related to the ratio of the areas of two triangles with the same height (or base), which states:

THEOREM 1

Areas of triangles of the same height (or base) are proportional to their bases (or heights).

This result can be found in Euclid's Elements Book VI (Heath, 1956) and its proof is quite trivial. However, we can use it to derive another simple and useful result, which states:

THEOREM 2

In Fig. 2 below, the ratio of the area of ΔACB to that of ΔDBC is equal to $AE : ED$.

This result follows readily from Theorem 1. If we draw the heights AF and DG in Fig. 2, then it is trivial to deduce this theorem by using the similarity property of ΔAEF and ΔDEG (Fig. 3).

In the next section, we will demonstrate how to use these results to prove the Routh Theorem.

3. The Routh Theorem

THEOREM 3 (The Routh Theorem)

In Fig 4, let the area of ΔABC be S_{ABC}. If the points D, E, F divide the sides of ΔABC such that $BD : DA = 1 : p$, $AE : EC = 1 : q$, $CF : FB = 1 : r$, then the area of ΔGHI is equal to

$$\frac{(pqr-1)^2}{(p+pq+1)(q+qr+1)(r+pr+1)} \times S_{ABC}.$$

PROOF. First, let us draw the line segment AG (Fig. 5).

Let the areas of ΔBDG, ΔDGA, ΔBGC and ΔGAC be S_{BDG}, S_{DGA}, S_{BGC} and S_{GAC}, respectively. By Theorem 1, we have $S_{DGA} = p \times S_{BDG}$. By Theorem 2, we have $S_{BGC} = q \times S_{BGA} = q(1+p) \times S_{BDG}$

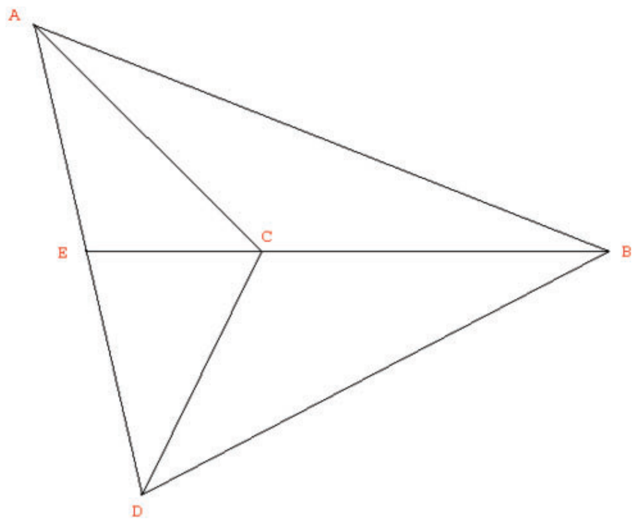

FIG. 2. Triangles with common base. (This figure appears in colour in the online version of Teaching Mathematics and its Application.)

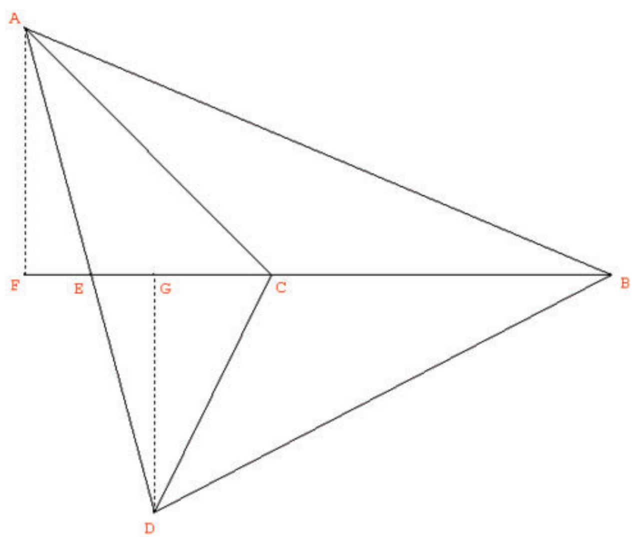

FIG. 3. Triangles with common base. (This figure appears in colour in the online version of Teaching Mathematics and its Application.)

and $S_{GAC} = p \times S_{BGC} = pq\,(1+p) \times S_{BDG}$. Since $S_{BDG} + S_{DGA} + S_{BGC} + S_{GAC} = S_{ABC}$, so we have

$$S_{BDG} \times [1 + p + q(1+p) + pq(1+p)] = S_{ABC}.$$

Hence,

$$S_{BDG} = \frac{S_{ABC}}{(1+p)(1+q+pq)} \quad \text{and} \quad S_{BGC} = \frac{q}{1+q+pq} \times S_{ABC}$$

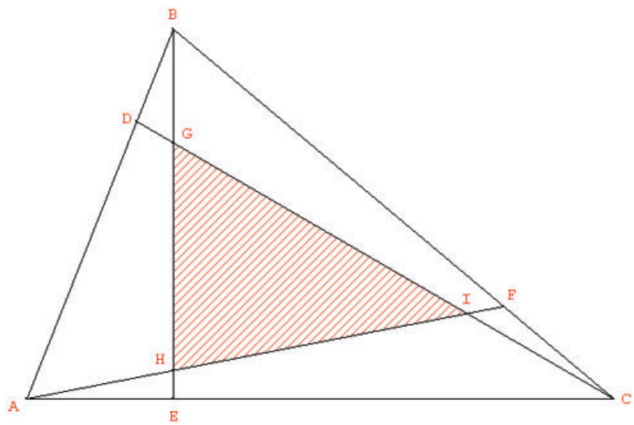

FIG. 4. The area of ΔGHI is determined by the Routh Theorem. (This figure appears in colour in the online version of Teaching Mathematics and its Application.)

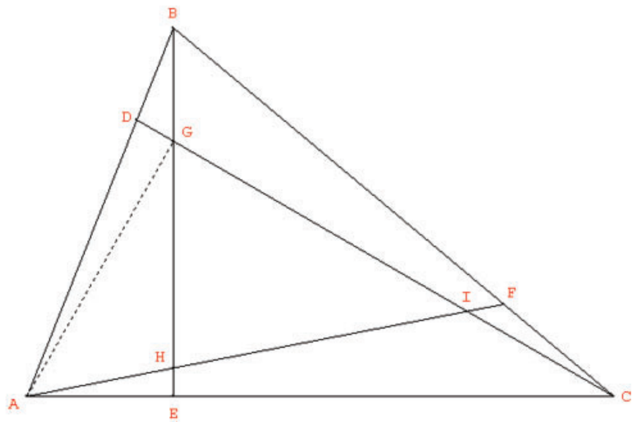

FIG. 5. The line segment AG is drawn inside the triangle. (This figure appears in colour in the online version of Teaching Mathematics and its Application.)

Similarly, by adding the line segments BI and HC, we have

$$S_{FIC} = \frac{S_{ABC}}{(1+r)(1+p+pr)} \quad \text{and} \quad S_{AIC} = \frac{p}{1+p+pr} \times S_{ABC},$$

$$S_{AHE} = \frac{S_{ABC}}{(1+q)(1+r+qr)} \quad \text{and} \quad S_{ABH} = \frac{r}{1+r+qr} \times S_{ABC}.$$

Hence,

$$\begin{aligned} S_{GHI} &= S_{ABC} - S_{BGC} - S_{AIC} - S_{ABH} \\ &= \left(1 - \frac{q}{1+q+pq} - \frac{p}{1+p+pr} - \frac{r}{1+r+qr}\right) \times S_{ABC} \end{aligned}$$

$$= \frac{(pqr)^2 - 2pqr + 1}{(1+q+pq)(1+p+pr)(1+r+qr)} \times S_{ABC}$$
$$= \frac{(pqr-1)^2}{(1+q+pq)(1+p+pr)(1+r+qr)} \times S_{ABC}\,.$$

It completes the proof.

4. Concluding remarks

In this article, we have demonstrated how to deduce the Routh Theorem by simple results in plane geometry, which can be introduced to students at secondary or undergraduate level. Finally, it is worthwhile to point out that the results below can be obtained by the use of the Routh Theorem:

(a) When $p=q=r=2$, $S_{GHI}=(8-1)^2 \times S_{ABC}/(1+2+4)^3 = 1/7 \times S_{ABC}$, which is the case of the original Feynman's Triangle problem.

(b) When $p=q=r$, we have $(p^3-1)^2 \times S_{ABC}/(1+p+p^2)^3 = (p-1)^2/(1+p+p^2)$.

(c) When the line segments CD, BE, AF are concurrent, $S_{GHI}=0$ and $pqr=1$, which is the so-called Ceva Theorem.

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