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A FIBONACCI GENERALIZATION: A LAKATOSIAN EXAMPLE

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This paper presents a generalization of the Fibonacci series which roughly followed a Lakatosian heuristic. Using this example, some general comments will be made regarding the processes of discovery and invention in mathematics, and its relevance to the history and philosophy of mathematics.

In this paper a generalization of the Fibonacci series which roughly followed a Lakatosian heuristic will be presented. This example illustrates the often important interplay between quasi-empirical investigation and logical analysis in the production of mathematical knowledge.

Afterwards, there will be a brief reflection not only on the role and limitations of quasi-empirical methods in relation to this particular example, but also in general in mathematical research. A personal model of how new discoveries are sometimes made in mathematics which incorporate these perspectives will also briefly be discussed. Lastly, some comments regarding the nature and philosophy of mathematics will be made.

DE JAGER'S PAPER

Some years ago, Tiekie de Jager, a gifted high school teacher at Rondebosch Boys' High School in Cape Town, South Africa gave the following problem to his Grade 9 pupils to explore:

Problem

Consider the following series:

(a) **1 + 1 + 2 + 3 + 5** + 8 + 13 + 21 + 34 + 55 + ...

(b) **1 + 1 + 2 + 3 + 5 + 8 + 13** + 21 + 34 + 55 + ...

In each case find the sum of the bold terms. What do you notice? Try your idea in some other cases.

This is the well-known Fibonacci series that the children already knew. In other words, they already knew that if we call the nth term T_n, each term could easily be constructed by the rule $T_n + T_{n+1} = T_{n+2}$. From the above calculations, the children then noticed that if the sum to *n* terms was called S_n, then the following pattern arose: $1 + S_n = T_{n+2}$.

Further Investigation

Upon asking the children to explore variations on this rule, one child suggested that they could perhaps try and find a similar series in which $1 + S_n = T_{n+3}$. With the aid of Tiekie, the children then found the following series:

1 + 1 + 1 + 2 + 3 + 4 + 6 + 9 + 13 + <u>19</u> + 28 + ...

which is formed by the following rule: $T_n + T_{n+2} = T_{n+3}$.

Soon the following series was also found:

1 + 1 + 1 + 1 + 2 + 3 + 4 + 5 + 7 + 10 + <u>14</u> +19 + ...

which is formed by the rule: $T_n + T_{n+3} = T_{n+4}$ and has the property that $1 + S_n = T_{n+4}$.

The next question was whether it was possible to find a series so that $1 + S_n = T_{n+1}$.

Soon the following series was found:

1 + 2 + 4 + 8 + <u>16</u> + 32 + 64 + ...

Generalization

This information was then summarized as follows in a table:

Rule	**Adding property**
$T_n + T_n = T_{n+1}$	$1 + S_n = T_{n+1}$
$T_n + T_{n+1} = T_{n+2}$	$1 + S_n = T_{n+2}$
$T_n + T_{n+2} = T_{n+3}$	$1 + S_n = T_{n+3}$
$T_n + T_{n+3} = T_{n+4}$	$1 + S_n = T_{n+4}$

which naturally led to the following generalization:

A series has the property $1 + S_n = T_{n+k+1}$, if and only if, it is generated by the rule $T_n + T_{n+k} = T_{n+k+1}$.

By this time several other classes of Tiekie de Jager had become involved in the investigation, and one of his top Grade 11 pupils, Shannon Kendal, eventually produced the following proof. It was based on the assumption of the following two statements (which follow automatically from the notation):

(a) $S_n = S_{n-1} + T_n$

(b) If $T_{n+k+1} = 1 + S_n$, then $1 + S_{n-1} = T_{n+k}$ (*n* becomes *n*-1)

Proof

$T_{n+k+1} = 1 + S_n$

<=> $T_{n+k+1} = 1 + S_{n-1} + T_n$ ... by (a)

$\Leftrightarrow \quad T_{n+k+1} = T_{n+k} + T_n$ ... by (b)

Tiekie presented this investigation at the 1989 National Convention of Mathematics and Science Teachers in Pretoria and later published it as part of a paper on Pattern Finding in **Pythagoras**, a South African mathematics education journal (De Jager, 1990).

COUNTER-EXAMPLES

After first reading through the above examples, the plausible generalization and convincing proof, I was initially inclined to accept the validity of the result and its proof without reservation. However, upon later testing it by specific examples to get a better feeling for the result, I found some problems with it as reported in De Villiers (1990). For example, after posing the question to find (construct) a series in which $1 + S_n = T_{n+3}$, the following series was produced by Tiekie and his students: 1 + 1 + 1 + 2 +3 + 4 + ... Of course, if we choose $T_1 = 1$, then according to $1 + S_n = T_{n+3}$ the fourth term T_4 must be $1 + S_1 = 2$. However, this does not necessarily imply that the second and third terms should necessarily both be 1. For instance if we choose both T_2 and T_3 equal to 2, we have according to $1 + S_n = T_{n+3}$, $1 + S_2 = 4 = T_{2+3} = T_5$ and $1 + S_3 = 6 = T_{3+3} = T_6$, giving us the series:

$$1 + 2 + 2 + 2 + 4 + 6 + 8 + ...$$

But is $T_n + T_{n+2} = T_{n+3}$ always true for this series as alleged? Unfortunately not, as $T_1 + T_3 = 1 + 2 = 3$ which is **not** equal to T_4. So here we have a *counter-example* to Kendal's theorem! Similarly, if we choose 2 as the first term, it is possible to construct the following series: 2 + 2 + 3 + 3 + 5 + 8 + 11 ... with the property $1 + S_n = T_{n+3}$, but $T_1 + T_2 = 2 + 3 = 5$ which is not equal to T_4. Similar counter-examples can easily be constructed for other values of k.

But what about the converse? Is it really true that a series which is constructed by the rule $T_n + T_{n+k} = T_{n+k+1}$ necessarily has the property that $1 + S_n = T_{n+k+1}$? Let us again consider the case $k = 2$, therefore the rule $T_n + T_{n+2} = T_{n+3}$. If we choose $T_1 = 1$ and $T_3 = 3$, we have $T_4 = 4$. If we now choose $T_2 = 2$, we have $T_5 = 6$, giving us the series: 1 + 2 + 3 + 4 + 6 + 9 + ... Is $1 + S_n = T_{n+3}$ for this series? Unfortunately the answer is again no, as $1 + S_1 = 2$ and that is not equal to T_4. Similar counter-examples can be constructed for other values of k.

Should we therefore simply dismiss Kendal's theorem as invalid? Or is there still something worthwhile saving? This was the challenge given to readers of **Pythagoras** in a letter (see De Villiers, 1990).

SCHUTTE'S RESPONSE

Schutte (1991) responded as follows: Firstly, he argued that my first counter-example was **not** valid. For example, he claimed that in order to make $T_1 + T_3$ equal to T_4, one had to make sure that $1 + S_0 = T_3$ since n has the value of 1 (according to the assumption $1 + S_{n-1} = T_{n+k}$). But $1 + S_0 \neq T_3$ since S_0 is undefined. Therefore it does not necessarily follow that $T_1 + T_3 = T_4$ because the conditions of the theorem are not met; thus my counter-example is invalid.

Secondly, he pointed out that my second counter-example (to show that $1 + S_n = T_{n+k+1}$ does not follow from $T_n + T_{n+k} = T_{n+k+1}$) is successful as the converse part of Kendal's proof is faulty. The converse part starts as follows:

$$T_{n+k+1} = T_{n+k} + T_n$$

$$\Rightarrow \quad T_{n+k+1} = 1 + S_{n-1} + T_n$$

Here T_{n+k} is replaced by $1 + S_{n-1}$. This can only be done if $T_{n+k+1} = 1 + S_n$ is already assumed as true. But this is exactly the formula which has to be proved! Hence the proof for the converse is circular and consequently invalid. The counter-example shows that the proof is incorrect.

DU TOIT'S RESPONSE

Du Toit (1991) similarly argued in regard to the forward implication that since T_0 and S_0 are both undefined, the theorem was limited to integer values of n greater than 1 (as indicated by my counter-example), and that to rectify the situation one only had to explicitly state this restriction for the forward implication.

Regarding my second counter-example: 1 + 2 + 3 + 4 + 6 + 9 ... which was developed according to the rule $T_n + T_{n+k} = T_{n+k+1}$ with $k = 2$, he pointed out that in this case $1 + S_n$ was not equal to T_{n+k+1} for any n. (As can easily be checked by the reader). However if one starts looking for a pattern in this series, one finds that $T_{n+k+1} - S_n$ gives a constant difference of 3 for all n; in other words, it is equal to T_3.

Next he considered $k = 4$. The first $k + 1$ terms can be chosen arbitrarily: 7 - 5 + 8 + 2 +20. Develop further according to the rule $T_n + T_{n+k} = T_{n+k+1}$; $T_6 = T_1 + T_5 = 27$; $T_7 = T_2 + T_6 = 22$; $T_8 = T_3 + T_7 = 30$; etc. This gives the series:

$$7 - 5 + 8 + 2 + 20 + 27 + 22 + 30 + 32 + 52 + \ldots$$

This gives:

$$S_1 + 20 = T_6$$

$$S_2 + 20 = T_7$$

$$S_3 + 20 = T_8$$

and since $T_5 = 20$, this gives the generalization: $T_{n+k+1} = S_n + T_{k+1}$.

He then gave the following reformulation of Kendal's theorem:

If T_n is the nth term and S_n is the sum to n terms of a series then for all $n > 1$:

$T_{k+1} + S_n = T_{n+k+1} \Leftrightarrow T_n + T_{n+k} = T_{n+k+1}$.

This was followed by a proof of the forward implication: If $T_{k+1} + S_n = T_{n+k+1}$, then for all $n > 1$, $T_n + T_{n+k} = T_{n+k+1}$, by adapting Kendal's method.

Proof

(a) $S_n = S_{n-1} + T_n$

(b) $T_{n+k+1} = T_{k+1} + S_n \Leftrightarrow T_{n+k} = T_{k+1} + S_{n-1}$ (n is replaced by n - 1).

$T_{n+k+1} = T_{k+1} + S_n$

$\Rightarrow$ $T_{n+k+1} = T_{k+1} + S_{n-1} + T_n$ ... from (a)

$\Rightarrow$ $T_{n+k+1} = T_{n+k} + T_n$ ... from (b)

The converse: If $T_n + T_{n+k} = T_{n+k+1}$, then for all n, $T_{k+1} + S_n = T_{n+k+1}$, was then proved using mathematical induction.

Proof

Assume it is true for $n = p$; therefore that the following is true:
If $T_p + T_{p+k} = T_{p+k+1}$ then $T_{k+1} + S_p = T_{p+k+1}$.

Now consider $n = p + 1$:

$$\begin{aligned} T_{p+1+k+1} &= T_{p+1} + T_{p+1+k} \\ &= T_{p+1} + S_p + T_{k+1} \\ &= S_{p+1} + T_{k+1}. \end{aligned}$$

Therefore if the statement is true for $n = p$, it is also true for $n = p + 1$. But if $T_1 + T_{1+k} = T_{1+k+1}$, then $S_1 + T_{1+k} = T_{1+k+1}$ ($T_1 = S_1$) and the statement is therefore true for $n = 1$; and therefore also for $n = 2$; etc.

THE MEANING OF S_0

In the November 1991 issue of **Pythagoras**, I again responded to Schutte and Du Toit's analysis as follows (De Villiers, 1991). In his letter, Schutte claimed that the following counter-example I gave for Kendal's original forward implication (if a series is constructed by the rule $1 + S_n = T_{n+k+1}$ then $T_n + T_{n+k} = T_{n+k+1}$) is invalid:

$1 + 2 + 2 + 2 + 4 + 6 + 8 + \ldots$ ($k = 2$)

Although it is true for this series that $T_2 + T_4 = T_5$, $T_3 + T_5 = T_6$, etc., I pointed out that it is not true for $n = 1$, since $T_1 + T_3 \neq T_4$.

The gist of Schutte's argument was that according to the assumption (b) in the original proof: $1 + S_{n-1} = T_{n+k}$, I should have ensured that $1 + S_0 = T_3$ since n is given the value of 1. But $1 + S_0$ cannot equal T_3 since S_0 is undefined, and therefore the conditions of the theorem are not met and the counter-example is invalid.

To this I responded by saying that using this very same argument it would also follow that 1 + S_0 in the original example 1 + 1 + 1 + 2 + 3 + 4 + ..., also cannot equal T_3 since S_0 is undefined. This is furthermore true for any **chosen** T_3, or in general for any chosen T_{k+1}, and therefore the logical consequence of this argument by Schutte would be that **no** series can ever be constructed according to the rule $1 + S_n = T_{n+k+1}$, since no satisfactory T_{k+1} can ever be chosen! Thus strictly according to Schutte's argument, Kendal's forward implication would become a totally contentless theorem ; ie. a theorem about a series which cannot be constructed.

The point that I therefore made was that S_0 actually becomes (or needs to become) **implicitly defined** in the construction of such a series, even though S_0 has no meaning (is undefined) in the traditional sense in terms of the sum of the first n terms. (An analogous example is the necessity to define $a^0 = 1$, although it at first appears a meaningless statement). For example for $k = 2$, $S_0 = 0$ in the original series 1 + 1 + 1 + 2 + 3 + 4 + ..., but for the two series 1 + 2 + 2 + 2 + 4 + 6 + 8 + .. and 2 + 2 + 3 + 3 + 5 + 8 + 11 + ... we respectively have $S_0 = 1$ and $S_0 = 2$. In fact, it follows directly from the assumption (b) in the original proof: $1 + S_{n-1} = T_{n+k}$ that in general $S_0 = T_{k+1} - 1$.

I then continued by asking the critical question: *why* the relationship is true for $n = 1$ for the original series 1 + 1 + 1 + 2 + 3 + 4 + ... (as well as the other examples given by Tiekie), but not for series like 1 + 2 + 2 + 2 + 4 + 6 + 8 + .. and 2 + 2 + 3 + 3 + 5 + 8 + 11 + ... To simply say that S_0 is "*undefined*" explains nothing at all (and in fact is misleading), since we have seen that S_0 becomes implicitly defined as $T_{k+1} - 1$ when constructing such a series. The explanation lies elsewhere, namely with the other assumption $S_n = S_{n-1} + T_n$ (assumption (a) in the original proof). If we set $n = 1$ in this assumption we have $S_0 = S_1 - T_1 = 0$. This assumption will therefore be true for $n = 1$ only if we choose T_{k+1} in such a way that S_0 becomes 0 in $S_0 = T_{k+1} - 1$; therefore T_{k+1} must be 1. In other words, if we choose $T_{k+1} \neq 1$ as in my two examples above (where $k = 2$), assumption (a) becomes false for $n = 1$ (but is still true for the other values of n); and therefore the conclusion that $T_1 + T_3$ must be equal to T_4 is also false.

The above analysis led me to the following generalization of the forward implication:

If T_n is the nth term and S_n is the sum to n terms of a series, then for all $n > 1$:

$C + S_n = T_{n+k+1} \Rightarrow T_n + T_{n+k} = T_{n+k+1}$

where C is any real number and $k \geq 0$.

Proof

(a) $S_n = S_{n-1} + T_n$ ... $n > 1$

(b) $T_{n+k+1} = C + S_n \Leftrightarrow T_{n+k} = C + S_{n-1}$ (n is replaced by n - 1).

$T_{n+k+1} = C + S_n$

$\Rightarrow$ $T_{n+k+1} = C + S_{n-1} + T_n$ ... from (a)

$\Rightarrow$ $T_{n+k+1} = T_{n+k} + T_n$ ... from (b)

In order to construct a series of this type, S_0 becomes implicitly defined as $S_0 = T_{k+1} - C$ (from assumption (b)). Note however that we still have from assumption (a) that $S_0 = S_1 - T_1 = 0$. Assumption (a) will therefore be true for $n = 1$ only if we choose T_{k+1} in such a way that S_0 becomes 0 in $S_0 = T_{k+1} - C$; therefore T_{k+1} must be equal to C. In other words, if we choose $T_{k+1} \neq C$, assumption (a) becomes false for $n = 1$ (but is still true for the other values of n); and therefore the conclusion that $T_1 + T_{k+1}$ must be equal to T_{k+2} also becomes false.

Example 1 (T_{k+1} $\boxed{\neq}$ C)

Suppose for $k = 2$ we arbitrarily choose $T_1 = 1$ and $C = -3$. Then from the rule $C + S_n = T_{n+k+1}$, it follows that $T_4 = -3 + 1 = -2$. If we now also arbitrarily choose $T_2 = 6$ and T_3 not equal to C, say $T_3 = -4$ (or equivalently choose $S_0 = -1$ from assumption (b)), we can construct the series:

$1 + 6 - 4 - 2 + 4 + 0 - 2 + 2 + 2 + 0 + 2 + 4 + \ldots$

Here we clearly have:

$T_2 + T_4 = 6 + (-2) = 4 = T_5$;

$T_3 + T_5 = -4 + 4 = 0 = T_6$;

$T_4 + T_6 = -2 + 0 = T_7$; etc.

Note however that since $T_{k+1} \neq C$, assumption (a) is not valid for $n = 1$, and therefore $T_1 + T_3 = 1 + (-4) = -3 \neq T_4$.

Example 2 (T_{k+1} = C)

Suppose for $k = 2$ we arbitrarily choose $T_1 = 1$ and $C = -3$. Then from the rule $C + S_n = T_{n+k+1}$, it follows that $T_4 = -3 + 1 = -2$. If we now also arbitrarily choose $T_2 = 6$ and T_3 equal to C, ie. $T_3 = -3$ (or equivalently choose $S_0 = 0$ from assumption (b)), we can construct the series:

$1 + 6 - 3 - 2 + 4 + 1 - 1 + 3 + 4 + 3 + 6 + 10 + \ldots$

Here we clearly have:

$T_1 + T_3 = 1 + (-3) = -2 = T_4$;

$T_2 + T_4 = 6 + (-2) = 4 = T_5$;

$T_3 + T_5 = -3 + 4 = 1 = T_6$; etc.

Note here that since $T_{k+1} = C$, assumption (a) is valid for $n = 1$, and therefore $T_1 + T_3 = T_4$.

Also note in the special case formulated by Du Toit, namely, that:
"If T_n is the nth term and S_n is the sum to n terms of a series, then for all $n > 1$:
$T_{k+1} + S_n = T_{n+k+1} \Leftrightarrow T_n + T_{n+k} = T_{n+k+1}$"
the restriction $n > 1$ (for the forward implication) is not necessary as it is valid for $n = 1$, since $C = T_{k+1}$; ie. this special case is valid for all n.

Alternative Proof for Converse

In conclusion, I also gave the following alternative proof for the converse (which to me personally was more explanatory).

The converse: If $T_n + T_{n+k} = T_{n+k+1}$, then for all n, $T_{k+1} + S_n = T_{n+k+1}$ where $k \geq 0$.

Proof

Firstly write the consecutive terms of the series as the following differences:

$$T_1 = T_{k+2} - T_{k+1}$$
$$T_2 = T_{k+3} - T_{k+2}$$
$$T_3 = T_{k+4} - T_{k+3}$$
$$.$$
$$.$$
$$.$$
$$T_{n-1} = T_{k+n} - T_{k+n-1}$$
$$T_n = T_{k+n+1} - T_{k+n}$$

Then adding up the left and right columns respectively, we find the desired result $S_n = T_{k+n+1} - T_{k+1}$ or $S_n + T_{k+1} = T_{n+k+1}$.

REFLECTION

The role of quasi-empirical testing

Let us now briefly reflect on the role of quasi-empirical testing in this example. Firstly we saw that it was the construction of four different series, and the observation of the underlying pattern, that led Tiekie de Jager and his students to formulating Kendal's theorem. This was followed up by the construction of a very convincing argument which appeared to validate the result.

Unfortunately we sometimes have a tendency to sit back and relax once a new theorem is proved and to rub our hands in satisfaction that its truth has been established. However as this example has shown, it is sometimes useful to check proven results by quasi-empirical testing as it may expose problems with our proof and/or formulation of the result. In this particular case, my counter-examples eventually not only led to identifying the circularity in the proof of the converse, but also in a precise formulation of a further generalization. In other words, pupils who exhibit a further need for empirical testing after a formal proof (as reported by Fischbein, 1982) should not be too harshly criticized.

Furthermore, Du Toit also first looked at special cases to find a pattern which led to his formulation and proof of a generalization of the converse (and forward implication). The subtle point that S_0 needs to be defined as $S_0 = T_{k+1} - C$ in order to construct a series according to the rule $C + S_n = T_{n+k+1}$ in the forward implication also only became apparent from the actual construction of such series. In other words, without the quasi-empirical experience of actually constructing such series according to the rule $C + S_n = T_{n+k+1}$, one could easily interpret S_0 simply as "*undefined*".

Schutte's rejection of my counter-example for the forward implication is in many respects similar to the technique of "*monster-barring in defence of the theorem*" described by Lakatos (1983). Since refutation by counter-example usually depends on the meaning of the terms involved, definitions are frequently proposed and argued about. In this particular case, it revolved around the meaning (definition) of S_0 as I pointed out in my last letter. A similar situation is described by Lakatos (1983:16) where, after the discovery of a counter-example to the Euler-Descartes theorem for polyhedra, the characters in his book then vehemently argue about whether to accept or reject the counter-example, for example:

> "*DELTA: But why accept the counter-example? We proved our conjecture - now it is a theorem. I admit that it clashes with this so-called 'counter-example'. One of them has to give way. But why should the theorem give way, when it has been proved? It is the `criticism' that should retreat. It is fake criticism. This pair of nested cubes is not a polyhedron at all. It is a* ***monster****, a pathological case, not a counter-example.*
>
> *GAMMA: Why not?* ***A polyhedron is a solid whose surface consists of polygonal faces****. And my counter-example is a solid bounded by polygonal faces.*
>
> *DELTA: Your definition is incorrect. A polyhedron must be a* ***surface****: it has faces, edges, vertices, it can be deformed, stretched out on a blackboard, and has nothing to do with the concept of 'solid'.* ***A polyhedron is a surface consisting of a system of polygons.***"

From the above extract, it is clear that refutation by counter-example in the Lakatosian model depends on the meaning of the terms involved and consequently definitions are frequently proposed and argued about.

The psychology of mathematical discovery and proof

What follows now is a personal model of how new discoveries are sometimes made in mathematics and is based on some of the explorations I have done in mathematics in the past, and will try to illustrate it in relation to the example we have just had. There is no intention however to present it as a model which encompasses the complex totality and rich diversity of mathematical discovery and proof.

Logically, mathematics is based upon the following fundamental axiom: *"Something is true (T), if and only if, it can be (deductively) proved (P)"* . However, from a psychological perspective, it is often more useful to represent it in the following equivalent, but different logical forms:

(a) the forward implication (T ⇒ P): if something is true, then it can be proved.

(b) the converse (P ⇒ T): if something has been proved, then it is true.

(c) the inverse (T' ⇒ P'): if something is false, then it cannot be proved.

(d) the contrapositive (P' ⇒ T'): if something cannot be proved, then it is false.

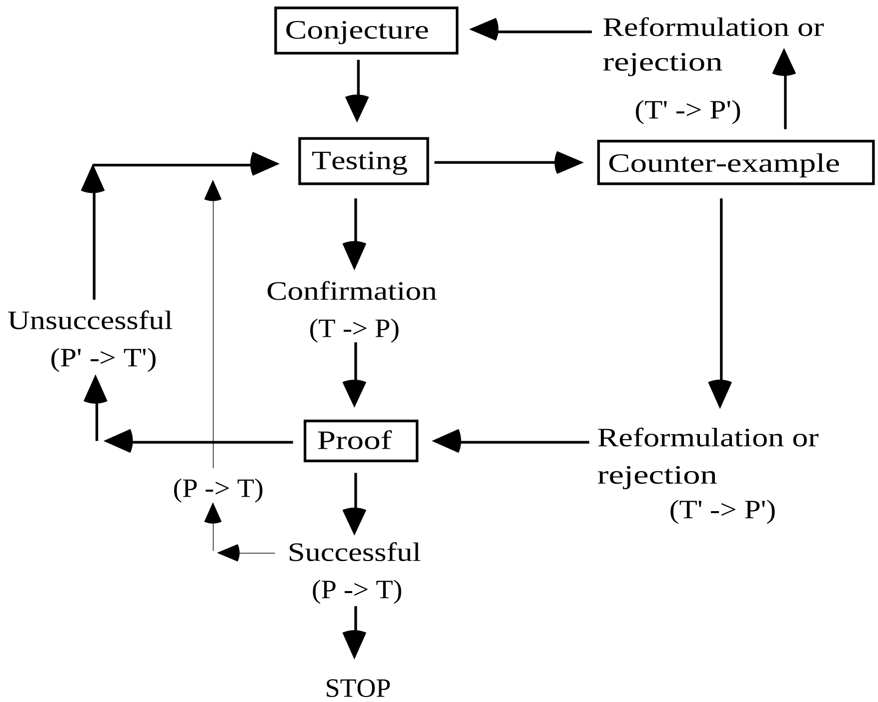

Figure 1

Unfortunately in textbooks and teaching only the converse (P ⇒ T) is usually conveyed; in other words, that we must first prove results, before we can accept them as true. However, in actual mathematical research as demonstrated in this paper, the forward implication (T ⇒ P), its inverse (T' ⇒ P') and contrapositive (P' ⇒ T') often play a far greater role in motivating and guiding our actions.

For example, suppose we inductively make a conjecture on the basis of some pattern in specific cases such as the four series right at the beginning. We might then start believing it to be true, which according to the forward implication then gives us the encouragement to start looking for a proof. However, if after a while we are not successful in producing a proof, we might start doubting the validity of the conjecture according to the contrapositive, and then consider some more cases, after which the whole process is of course repeated. If the conjecture is not supported by these additional cases, we either reject it as false according to the inverse (and longer even bother trying to prove it) or have to reformulate/refine it. Furthermore, even though the converse logically implies that once it has been proved, then it must be true for all cases, additional testing as shown by dotted line may be valuable in exposing problems in one's proof and/or formulation of the result (as was so clearly shown in this example).

This process of conjecturing, testing, refuting, proving and reformulating can sometimes (definitely not always!) go through several cycles and is represented in Figure 1. In the above model, conviction is neither seen as the exclusive prerogative of proof nor the only function of proof as that of verification/conviction. To the contrary, as shown in this example, conviction based on quasi-empirical exploration often precedes proof and is probably far more frequently a prerequisite for the finding of a proof (compare De Villiers, 1997).

Of course, this immediately raises the question of why do we still feel a need to prove results for which we have substantial quasi-empirical support. There are many reasons such as the very real possibility that the observed pattern between the series may break down, our need to understand (and explain) why the result is true, the intellectual challenge of constructing a proof, etc. (compare De Villiers, 1990). The first two reasons above highlight two very serious shortcomings of quasi-empirical exploration, namely, (1) that quasi-empirical testing provides no guarantee for the general validity of the observed pattern, and (2) although checking more and more cases may succeed in increasing one's level of conviction, mere checking seldom provides any insight into why the pattern is true; it simply confirms that it is true.

The nature and philosophy of mathematics

Fallibilism has become very popular amongst many mathematics educators in recent years and can roughly be described as the view that mathematics is fallible, contestable and just as subjective as other areas of knowledge. In some quarters (eg. Borba & Skovsmose, 1997), fallibilism has also assumed the role of a political ideology which is opposed to the traditional "ideology of absolutism and certainty". To a fallibilist no knowledge (including mathematical knowledge) is stable; it is constantly in a state of change, being challenged, refuted and replaced. The fallibilist, therefore, strongly believes that the Lakatos model of heuristic refutation provides an adequate description of the nature of mathematics, as well as its discovery and invention. Fallibilist ideas also feature strongly in the philosophy of "*social constructivism*" proposed by Ernest (1991).

Although the Fibonacci example discussed here gives partial support to such a view, I can probably give many more examples from my own mathematical research which did not follow this pattern. As pointed out by Hanna (1995; 1996), as well as the prominent mathematician John Conway in Sept 1995 on the Mathematics Forum on the World Wide Web, it is not difficult to cite many historical cases where the mathematical development was radically different from the heuristic refutation described by Lakatos. As Conway pointed out:

> "*It is misleading to take this example (Euler-Descartes) as typical of the development of mathematics. Most mathematical theorems do get proved, and stay proved; the original proof may not be quite satisfactory according to later standards of proof, but that is a fairly trivial matter.*"

In fact, we must remember that Lakatos himself generalized his philosophy from the historical analysis of only two cases, namely, the Euler-Descartes theorem and Cauchy's primitive conjecture for uniform convergence. As mathematicians, we ought to know the dangers from generalizing too quickly from just a few cases! It would appear as if Fallibilism is turning the exception into the rule.

Furthermore if we look carefully at the example we have discussed here: from its initial generalization, formulation, primitive proof, counter-examples, further exploration and refinement, we find that the result itself actually underwent very little change; and its truth was never seriously in doubt. The problem was more in carefully sorting out the underlying logic and precise conditions of the theorem. The same applies to the Euler-Descartes theorem where the formula $V - E + F = 2$ was never seriously questioned; the real problem was to rigorously define the concept polyhedra and to develop a satisfactory general proof that would cover all cases.

A general weakness of Fallibilism and the Lakatosian model is also that these views could very easily degenerate into a kind of mathematical nihilism - everything is doubtful and fallible. If that were the case, why then study or do research in mathematics if everything was simply going to be disproved or falsified tomorrow? To me personally, mathematics would then certainly not be something sensible to engage in.

Fallibilists also frequently appear to confuse the **heuristic** counter-examples of Lakatos with **logical** counter-examples; in other words, they seem to interpret them as if they completely invalidate or falsify the original statement. In the words of one of the fictional students of Lakatos (1983: 86) these counter-examples are mostly not "*logical counter-examples, since they are after all not inconsistent with the conjecture in its intended interpretation*", and instead calls them "*heuristic counter-examples since they spur the growth of knowledge.*" If we carefully look back at the counter-examples I gave to Kendal's theorem (particularly the one to the forward implication), then it becomes clear that they are more heuristic than logical; in other words, they did not completely refute the original statement, but simply allowed for refinement and further generalization.

Furthermore, it is a highly questionable assumption of Fallibilism that this process of proof and heuristic refutation can in principle carry on indefinitely. Historical evidence, and this particular Fibonacci example, strongly indicate that there are limits to this process. For example, although it is quite likely that Kendal's theorem may be reformulated, further generalized, that alternative proofs may be developed, etc., it has in the final version reached a point where we can now say with certainty, since the proofs are valid, that the result is true, and that no further (logical) counter-examples are possible. Any philosophy of mathematics which does not acknowledge that we can attain this kind of certainty in mathematics is unrealistic.

In attacking the Formalist, Platonist, Intuitionist and other philosophical positions on the nature of mathematics, a radical fallibilist view is in danger of throwing out the baby with the bathwater. The wonder and mystery of mathematics is that despite our often fumbling in the dark, we can at some stage reach a point where we have proved and clarified ambiguities, and when we can say with certainty that something is true. Such an experience is liberating, illuminating and empowering. At the same time it is also extremely humbling that despite our human fallibility and imperfections, we can at some stage arrive at mathematical knowledge that is certain and infallible. To deny this, is to deny the mathematical reality of meaning making experienced by countless practising mathematicians.

Although radical fallibilism (and social constructivism) denies the existence of mathematics as an independent, objective reality, the example discussed here actually also tends to support a Platonistic view of mathematics, namely, that mathematical objects like the Fibonacci series can have definite (uncontestable and objective) properties (eg. like Kendal's theorem) that we may or may not be able to discover. They are not mere social or cultural conventions, but appear to pre-exist independently to the consciousness of any one person or culture. Such a view is often necessary for research in mathematics, as it suggests that there may still be thousands of patterns and properties waiting out there to be discovered by any adventurous person. (The existence of a journal called the **Fibonacci Quarterly** which is entirely devoted to the rich variety of patterns and properties contained in the Fibonacci series clearly supports such a view). The mathematician Connes for example describes a mathematician as "*an explorer setting out to discover the world*" (see Tahta, 1996:19).

In conclusion, we should acknowledge that mathematics is one single, very complex phenomenon, and that the Platonist, Formalist, Intuitionist, Fallibilist, Socio-political and other views of it, all complement rather than oppose each other, since each contain an element of truth by providing a valuable perspective from a certain angle (compare Davis & Hersh, 1983:358-359). The danger lies in not recognizing the value of each of these different views and becoming dogmatically or ideologically tied to a single, narrow perspective.

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