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A FIBONACCI GENERALIZATION

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It is well known that the sum of any ten numbers of a Fibonacci sequence is divisible by 11. For example, starting with 11, 15 and proceeding to 26, 41, 67, 108, 175, 283, 458, 741, the sum of these ten terms is 1925 which is divisible by 11, the quotient being 175, the seventh member of the set of ten successive terms. This can be proved to hold in general, for if we start with any two numbers a, b, the successive terms are a + b, a + 2b, 2a + 3b, 3a + 5b, 5a + 8b, 8a + 13b, 13a + 21b, 21a + 34b, the sum of which is 55a + 88b which on being divided by 11 gives a quotient of 5a + 8b, the seventh term of the set of ten terms.

This curious property might lead one to speculate on the possibility of having various sets of successive terms of a generalized Fibonacci sequence divisible by some common quantity. To analyze the situation let us start with terms $T_1 = a$, $T_2 = b$, the usual Fibonacci relation.

$$T_{n+1} = T_n + T_{n-1}$$

Thus $T_3 = a + b$, $T_4 = a + 2b$, $T_5 = 2a + 3b$, $T_6 = 3a + 5b$, $T_7 = 5a + 8b$, etc. It appears that the coefficients of a and b are Fibonacci numbers from the sequence $(1, 1, 2, 3, 5, 8, 13, \cdots)$ with the general formula being

$$T_n = F_{n-2}a + F_{n-1}b$$

However, what we are considering is the sum of a certain number of terms of this sequence. We want to find what:

$$\sum_{k=1}^{n} T_k = T_1 + T_2 + T_3 + \cdots + T_n$$

equals in terms of a, b, and the Fibonacci numbers. Now it will be observed that the coefficients of a in the summation are the Fibonacci numbers

$1, 1, 2, 3, \cdots, F_{n-3}, F_{n-2}$ with an extra 1, so that by the usual formula for the sum of the Fibonacci numbers, the coefficient of a in the sum must be F_n; the coefficient of b is simply the sum of the Fibonacci numbers up to and including F_{n-1}, so that this coefficient is $F_{n+1} - 1$. Thus

$$\sum_{k=1}^{n} T_k = F_n a + (F_{n+1} - 1)b .$$

We are looking for a common factor of this sum, no matter what values a and b may have. Thus, we seek common factors of F_n and $F_{n+1} - 1$. In the case $n = 10$, $F_{10} = 55$, $F_{11} - 1 = 88$, so that the common factor is 11. A little experimentation leads to the following table. We begin with 10 and go to higher values.

55	11 · 5
89-1	11 · 8
144	8 · 18
233-1	8 · 29
377	29 · 13
610-1	29 · 21
987	21 · 47
1597-1	21 · 76
2584	76 · 34
4181-1	76 · 55
6765	55 · 123
10946-1	55 · 199
17711	199 · 89
28657-1	199 · 144

It appears that we have two cases. When n is of the form $4k + 2$, the common factor is a Lucas number and the quotients are successive Fibonacci numbers; while if n is of the form $4k$, the common factor is a Fibonacci number and the quotients successive Lucas numbers. More precisely, the intuitive relations would seem to be as follows:

$$\sum_{j=1}^{4k+2} T_j = F_{4k+2}a + (F_{4k+3} - 1)b$$
$$= L_{2k+1}F_{2k+1}a + L_{2k+1}F_{2k+2}b$$
$$= L_{2k+1}(F_{2k+1}a + F_{2k+2}b)$$
$$= L_{2k+1}T_{2k+3}$$

and for the other case:

$$\sum_{j=1}^{4k} T_j = F_{4k}a + (F_{4k+1} - 1)b$$

$$= L_{2k}F_{2k}a + F_{2k}L_{2k+1}b$$
$$= F_{2k}(L_{2k}a + L_{2k+1}b)$$
$$= F_{2k}(F_{2k-1}a + F_{2k}b + F_{2k+1}a + F_{2k+2}b)$$
$$= F_{2k}(T_{2k+1} + T_{2k+3})$$

The formula $F_{2n} = F_nL_n$ is well known. There are two other formulas $F_{4n+3} - 1 = L_{2n+1}F_{2n+2}$ and $F_{4n+1} - 1 = L_{2n+1}F_{2n}$ which need to be justified. We first verify them for small values of n. For $n = 0$, the first formula gives $F_3 - 1 = 2 - 1$ or 1 and $L_1F_2 = 1 \cdot 1$ and hence is 1 as well. For $n = 1$, the first formula has on the left $F_7 - 1 = 13 - 1$ or 12 and on the right, $L_3F_4 = 4 \cdot 3$ or 12. Thus the first formula holds for small values of n. Similarly, the second can be verified for these small values. We now assume that the various formulas hold up to F_{4n+2} and that $F_{2n} = F_nL_n$ holds in general. Then

$$F_{4n+1} - 1 = L_{2n+1}F_n$$
$$\underline{F_{4n+1} \quad = L_{2n+1}F_{2n+1}}$$

Adding $$F_{4n+3} - 1 = L_{2n+1}F_{2n+2}$$
Then $$\underline{F_{4n+4} \quad = F_{2n+2}L_{2n+2}}$$

Adding $$F_{4n+5} - 1 = F_{2n+2}L_{2n+3}$$

Hence, since these formulas hold for small values of n as shown, it follows that they can be proved to hold for any value of n by reason of mathematical induction. Thus the intuitive formulas are seen to hold in general.

As a numerical illustration of these formulas, consider the series starting with a = 8, b = 11.

k	T_k	ΣT_k	Factorization	In Symbols
1	8	8		
2	11	19	1 · 19	L_1T_3
3	19	38		
4	30	68	1 · 68	$F_2(T_3 + T_5)$
5	49	117		
6	79	196	4 · 49	L_3T_5
7	128	324		
8	207	531	3 · 177	$F_4(T_5 + T_7)$
9	335	866		
10	542	1408	11 · 128	L_5T_7
11	877	2285		
12	1419	3704	8 · 463	$F_6(T_7 + T_9)$
13	2296	6000		
14	3715	9715	29 · 335	L_7T_9
15	6011	15726		
16	9726	25452	21 · 1212	$F_8(T_9 + T_{11})$
17	15737	41189		
18	25463	66652	76 · 877	L_9T_{11}
19	41200	107852		
20	66663	174515	55 · 3173	$F_{10}(T_{11} + T_{13})$
21	107863	282378		
22	174526	456904	199 · 2296	$L_{11}T_{13}$
23	282389	739293		
24	456915	1196208	144 · 8307	$F_{12}(T_{13} + T_{15})$
25	739304	1935512		
26	1196219	3131731	521 · 6011	$L_{13}T_{15}$
27	1935523	5067254		
28	3131742	8198996	377 · 21748	$F_{14}(T_{15} + T_{17})$

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