A simple proof of an interesting Fibonacci generalization

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It is reasonably well known that the ratios of consecutive terms of a Fibonacci series converge to the golden ratio. This note presents a simple, complete proof of an interesting generalization of this result to a whole family of 'precious metal ratios'.

1. Introduction

A beautiful property of the Fibonacci series is that the ratios of consecutive terms converge to the golden ratio. By generalizing the recursive formula $T_n + T_{n+1} = T_{n+2}$ for a Fibonacci series to the general formula $T_n + T_{n+k} = T_{n+k+1}$, where $k = 1,2,\ldots,$ De Villiers [1] made the interesting discovery that for each member of this family of series, the ratios of consecutive terms converge to the positive roots of $x^{k+1} - x^k - 1 = 0$. However, based on the assumption that $\lim_{n\to\infty} (T_{n+k+1}/T_{n+k})$ exists, only a partial proof to this result was given.

De Villiers [2] suggested a simple proof for the case where k is odd, with the suggestion that it could be generalized to also cover the case where k is even. What follows is a generalization of this approach, and provides a complete proof of the result.

2. Some preliminaries

From the equation $x^{k+1} = x^k + 1$ (1) we can deduce that

$$x^k(x-1) = 1 \rightarrow x^k = \frac{1}{x-1}$$

and therefore, to solve equation (1) is to solve the system

$$\left\{ \begin{array}{l} f(x) = \dfrac{1}{x-1} \\ g(x) = x^k \end{array} \right\} \tag{2}$$

that is to say, to find the intersection of the curves defined in (2) for $k = 1,2,\ldots$

If k is even, the graph of $g(x) = x^k$ is a curve of parabolic type, and the intersection of the two curves is given in figure 1.
Then, system (2) has only one real solution $x = M > 1$, which approaches 1 as the value of k increases.

If k is odd, the graph of $g(x) = x^k$ is a curve of cubic type, and the intersection of the two curves is given in figure 2.
Note that system (2) admits only two real roots M and λ such that $M > 1$ and $-1 < \lambda < 0$. These solutions tend to 1 and -1 respectively when k increases.

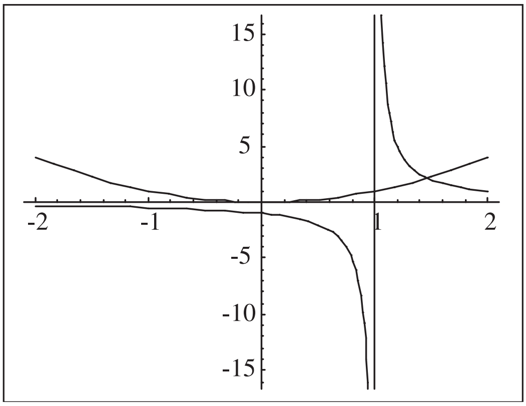

Figure 1.

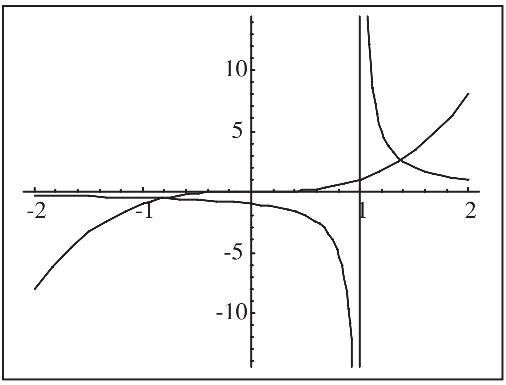

Figure 2.

The other roots are simple complex numbers and the modulus lie between $|\lambda|$ and M if k is an odd number and between $M/2$ and M if k is even.

In short: the roots of equation (1) are one positive real number M and k complex solutions λ_j of the form $\lambda_j = a_j + ib_j$ in such form that if k is odd, one of these roots lacks an imaginary part and its real part is negative. These k complex roots can be expressed in exponential form as $\lambda_j = r_j e^{i\varphi_j}$,where

$$r_j = \sqrt{a_j^2 + b_j^2} \quad \text{and} \quad tg\varphi_j = \frac{b_j}{a_j}$$

and such that $M > r_j$ for $j = 1,2,\ldots k$.

Theorem. If T_n is the nth term of a sequence with the property $T_n + T_{n+k} = T_{n+k+1}$, then for $k \geq 0$

$$\lim_{n\to\infty} \frac{T_{n+k+1}}{T_{n+k}} = M$$

where M is the positive root of $x^{k+1} - x^k - 1 = 0$.

Proof. The preceding sequence is equivalent to $T_n = T_{n-1} + T_{n-k-1}$ (it is enough to substitute $n+k+1$ by n) and this one is a difference equation which characteristic equation is $x^n = x^{n-1} + x^{n-k-1}$ or, that is the same, $x^{k+1} = x^k + 1$. Taking into account the discussion in the preceding section, we will have that the solution of the difference equation is (see [3])

$$T_n = a_1 M^n + \sum_{j=2}^{k+1} a_j \lambda_j^n \tag{3}$$

On the other hand,

$$\lim_{n\to\infty} \frac{T_{n+k+1}}{T_{n+k}} = \lim_{n\to\infty} \frac{T_n}{T_{n-1}}$$

so, taking into account formula (3) it is

$$\lim_{n\to\infty} \frac{T_n}{T_{n-1}} = \lim_{n\to\infty} \frac{a_1 M^n + \sum_{j=2}^{k+1} a_j \lambda_j^n}{a_1 M^{n-1} + \sum_{j=2}^{k+1} a_j \lambda_j^{n-1}} = \lim_{n\to\infty} \frac{a_1 + \sum_2^{k+1} a_j (\lambda_j/M)^n}{a_1(1/M) + \sum_2^{k+1} a_j (\lambda_j/M)^{n-1}(1/M)}$$

$$= \lim_{n\to\infty} \frac{a_1 + \sum_2^{k+1} a_j (r_j/M)^n e^{in\theta_j}}{(a_1/M) + \sum_2^{k+1} a_j (r_j/M)^{n-1}(e^{i(n-1)\theta_j}/r_j)} = \frac{a_1}{a_1/M} = M$$

because, since $M > r_j$,

$$\lim_{n\to\infty} \left(\frac{r_j}{M}\right)^n = 0.$$

References

[1] de Villiers, M., 2000, *Int. J. Math. Educ. Sci. Technol.*, **31,** 464.
[2] de Villiers, M., 2002, Private communication.
[3] Elaydi, S. N., 1999, *An Introduction to Difference Equations*, 2nd edn (New York: Springer-Verlag).

The inscribed sphere of an *n*-simplex

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The centre and radius of the inscribed n-dimensional sphere of an n-simplex are derived using elementary linear algebra.

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