A Forgotten Theorem for Triangle Similarity?

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Similarity is an important and fundamental concept in mathematics. In the real world it has many practical applications such as land surveying and the creation of maps and building plans. Within geometry it can be used to prove the theorem of Pythagoras as well as several circle geometry theorems.

The following two conditions for triangle similarity are well-known, and appear as theorems in current South African textbooks as well as in many high school textbooks around the world.

Theorem 1: If the angles of two triangles are correspondingly equal, then the corresponding sides of the two triangles are proportional, and the triangles are similar.

Theorem 2: If the corresponding sides of two triangles are proportional, then the angles of the two triangles are correspondingly equal, and the triangles are similar.

Less well known is the following theorem which is not currently prescribed in the South African curriculum but is often very useful in problem-solving and proving results about similarity:

Theorem 3: If two pairs of sides of two triangles are proportional, and the correspondingly included angles between these pairs of sides are equal, then the two triangles are similar[5,6].

Another important property of similarity that learners are perhaps not made aware of enough is that if two polygons X and Y are similar, then the ratios between the pairs of sides (or diagonals) of the polygon X equal the ratios between the corresponding pairs of sides (or diagonals) of polygon Y. For example, if two parallelograms X and Y are similar, and the long side of parallelogram X is twice that of its short side, then that it is also true for the ratio between the long and short side of parallelogram Y.

Specifically in the case when two triangles ABC and PQR are similar, the following ratios also hold as shown in Figure 1:

$$\frac{AB}{BC}=\frac{PQ}{QR}\ ;\ \frac{AB}{AC}=\frac{PQ}{PR}\ ;\ \frac{BC}{AC}=\frac{QR}{PR}$$

However, as shown by the measurement of the ratios in Figure 1, it is important to note that these three pairs of ratios are not necessarily all equal to each other. This 'inner-side-ratio' property of course follows directly from the similarity of the two triangles. For example, since triangles ABC and PQR are similar, we have corresponding sides in proportion. So $\frac{AB}{PQ}=\frac{BC}{QR}\Rightarrow\frac{AB}{BC}=\frac{PQ}{QR}$ etc.

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[5] The Siyavula site gives a proof this theorem in their similarity module at: https://www.siyavula.com/read/maths/grade-12/euclidean-geometry/08-euclidean-geometry-05

[6] In international high school textbooks, Theorem 1 is often indicated as AAA (or simply AA since having two corresponding angles equal is sufficient for the two triangles to be similar), Theorem 2 as SSS, and Theorem 3 as SAS.

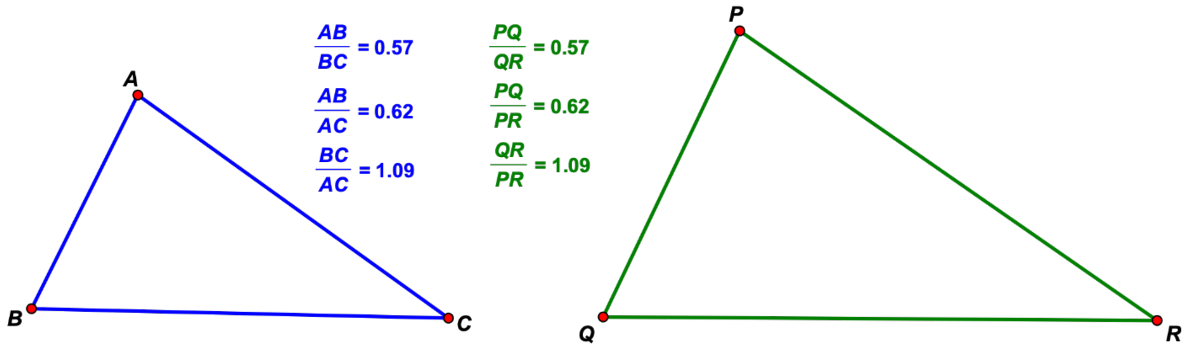

FIGURE 1

This 'inner-side-ratio' property of triangle similarity forms the fundamental basis of trigonometry. For example, as shown in Figure 2, all right triangles A_nBC_n with $\angle B = 90°$, and with say $\angle C_n = \theta$, are similar, since two corresponding angles are equal. Hence, the ratios of the sides $\frac{AB}{BC}$, $\frac{AB}{AC}$ and $\frac{BC}{AC}$ of all right triangles for a given angle θ are constant, and this gives us the three basic trigonometric ratios, namely, tangent, sine & cosine.

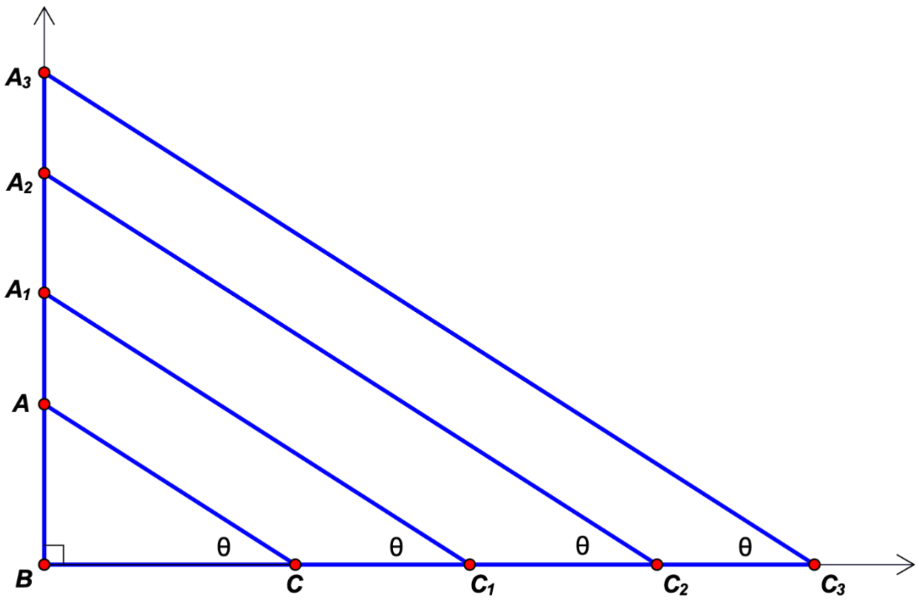

FIGURE 2

Recently I idly wondered about the converse, i.e. that if the ratios $\frac{AB}{BC} = \frac{PQ}{QR}$; $\frac{AB}{AC} = \frac{PQ}{PR}$; $\frac{BC}{AC} = \frac{QR}{PR}$ hold for two triangles ABC and PQR, would that necessarily also imply that the triangles are similar? Since I'd never before seen (or can't remember seeing) this result mentioned anywhere in a geometry textbook, I was skeptical about it being true. So I decided to first test it with a dynamic sketch, which would either give me a counter-example or confirm its truth.

Here are the detailed steps I used to construct a dynamic test of the converse with the aid of Sketchpad, but users of other dynamic geometry software packages should easily be able to duplicate the steps.

Step 1: Construct a dynamic ΔABC and a dynamic line segment QR.

Step 2: Measure the ratios $\frac{AB}{BC}$ and $\frac{AC}{BC}$.

Step 3: Mark ratio $\frac{AB}{BC}$ as a 'Scale Factor', and mark Q as a 'Center', then 'Dilate' the point R from Q as centre with the marked ratio $\frac{AB}{BC}$. With Q as centre draw a circle with radius QR' where R' is the image of the preceding dilation[7].

Step 4: Mark ratio $\frac{AC}{BC}$ as a 'Scale Factor', and mark R as a 'Center', then 'Dilate' the point Q from R as centre with the marked ratio $\frac{AC}{BC}$. With R as centre draw a circle with radius $Q'R$ where Q' is the image of the preceding dilation.

Step 5: Construct the intersection of the two circles, and label one of the intersections as P. (We ignore the other intersection, say P', since $\Delta P'QR$ is congruent to ΔPQR).

The completed construction is shown in Figure 3 (only part of the circle with centre R is shown). The reader is now invited to explore a dynamic sketch of this construction at:
http://dynamicmathematicslearning.com/forgotten-similarity-theorem.html

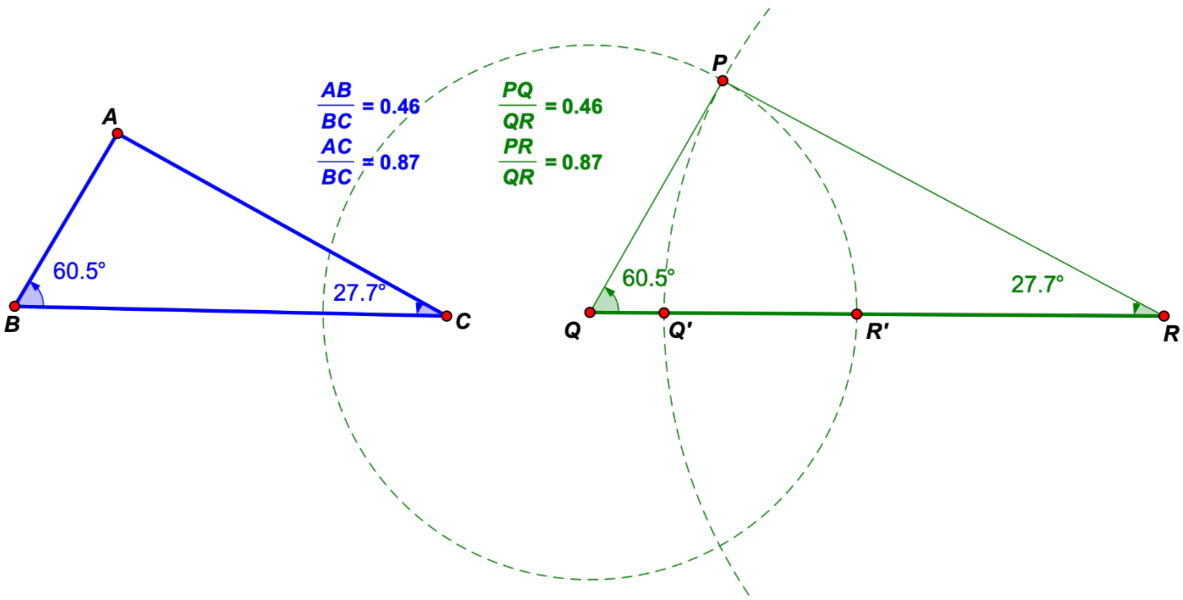

FIGURE 3

Clearly ΔPQR has been constructed so that $\frac{PQ}{QR} = \frac{AB}{BC}$ and $\frac{PR}{QR} = \frac{AC}{BC}$, but as shown in the figure, the measurements of two corresponding angles are equal, and hence ΔPQR is similar to ΔABC; in other words, it is a condition for similarity, and a theorem! The reader is invited to test the veracity of this conclusion by dragging the dynamic sketch at the URL given above.

[7] A dilation is a transformation that enlarges or reduces the size of a figure or a length by a fixed scale factor.

We can now formulate the theorem more precisely as follows:

THEOREM:

If for two triangles ABC and PQR any two pairs of ratios hold from the following three pairs of corresponding ratios, $\frac{AB}{BC}=\frac{PQ}{QR}$, $\frac{BC}{AC}=\frac{QR}{PR}$ and $\frac{AB}{AC}=\frac{PQ}{PR}$, then $\Delta ABC|||\Delta PQR$.

The proof follows from the construction and is quite straight forward.

PROOF:

Assume that for the two triangles ABC and PQR the following two ratios hold:

(1) $\frac{AB}{BC}=\frac{PQ}{QR}$ and

(2) $\frac{AB}{AC}=\frac{PQ}{PR}$

Dividing ratio (1) by ratio (2) gives:

$$\frac{AB}{BC}\times\frac{AC}{AB}=\frac{PQ}{QR}\times\frac{PR}{PQ} \Rightarrow \frac{AC}{BC}=\frac{PR}{QR} \Rightarrow \frac{QR}{BC}=\frac{PR}{AC}$$

But from ratios (1) and (2) we respectively have the following:

$$\frac{QR}{BC}=\frac{PQ}{AB} \text{ and } \frac{PR}{AC}=\frac{PQ}{AB}$$

Hence, the corresponding sides of the two triangles are in proportion, and from Theorem 2 it follows that $\Delta ABC|||\Delta PQR$. In the same way for the selection of any two of the other possible pairs of ratios, it can be shown that the sides of the two triangles would be proportional. This then completes the proof.

CONCLUDING REMARK

The result in this paper can be used as the basis of an investigation for high school learners to explore and consolidate a deeper understanding of triangle similarity.

Since this triangle similarity condition (and theorem) is so easy to prove, it seems unlikely that it is original. More likely is that it has been forgotten over time. It is clearly not well-known, and it seems surprising that it does not appear in any of the geometry textbooks available to me.