# Further reflections on a particular hexagon 

## Michael de Villiers



Michael de Villiers has worked as mathematics and science teacher and researcher at institutions across the world. From 1983-1990, he was at the University of Stellenbosch, and from 1991-2016 part of the University of Durban-Westville (the University of KwaZulu-Natal from 2005). Though retiring in 2016, he has held an honorary position at the University of Stellenbosch since then. He was editor of Pythagoras, the research journal of the Association of Mathematics Education of South Africa, and was vice-chair of the SA Mathematics Olympiad (SAMO) from 1997-2016, and is still Coordinator of the Senior Problem Solving Committee for SAMO. His main research interests are Geometry, Proof, Applications and Modeling, Problem Solving, and the History of Mathematics.
Homepage: http://dynamicmathematics-learning.com/homepage4.html
Dynamic Geometry Sketches: http://dynamic-mathematicslearning.com/JavaG
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#### Abstract

This paper follows on a previous paper about a particular hexagon and proves additional properties. For example, proving that the hexagon in question is tangential, i.e. has an incircle, formulating \& proving a converse, as well as exploring the conditions under which the hexagon becomes cyclic. Generalizations to particular $2 n$-gons are included.


## Introduction

In a recent paper by De Villiers \& Hung (2022) some concurrency, collinearity \& other properties of a hexagon $A B C D E F$ with $A B=B C, C D=D E, E F=F A$, and $\Varangle A=\Varangle C=\Varangle E=$ $\theta$ were explored. However, shortly after publication the following additional properties were discovered upon 'looking back' at the results in the style of Pólya (1945). These additional properties should also be of interest not only to talented mathematics olympiad students, but since the proofs are quite elementary, possibly also suitable as enrichment for average high school geometry classes.

## Incircle

Since the main diagonals of the hexagon above are concurrent, as proven in De Villiers \& Hung (2022), it was obvious from the converse of Brianchon's theorem that this particular hexagon had an inscribed conic. Somewhat surprisingly though, it turns out on further investigation that the inscribed conic is a circle! This gives us the first additional theorem below. An interactive dynamic geometry sketch for this result, and those further on, is available for the reader to explore at: http://dynamicmathematicslearning.com/further-hexagon-propert ies.html


Figure 13: Incircle of hexagon

## Theorem 1

Given a hexagon $A B C D E F$ with $A B=B C, C D=D E, E F=F A$, and $\Varangle A=\Varangle C=\Varangle E$, then $A B C D E F$ has an incircle.
Proof. Note that the angle bisectors of the angles at $B, D$ and $F$ are concurrent at the circumcentre, $Q$, of $\triangle A C E$. Hence, to prove the existence of an incircle, it suffices to show that the angle bisectors of the angles at $A, C$ and $E$ are also concurrent at $Q$. Connect $A, C$ and $E$ with $Q$. Now note that $A B C Q, A F E Q$ and $C D E Q$ are kites. Therefore, $\Varangle B A Q=\Varangle B C Q, \Varangle F A Q=\Varangle F E Q$ and $\Varangle D C Q=\Varangle D E Q$. But it is given $\Varangle B C Q+\Varangle D C Q=\Varangle F E Q+\Varangle D E Q$. Therefore, $\Varangle B C Q=\Varangle F E Q$, which implies that $\Varangle B A Q=\Varangle F A Q$, and there $A Q$ bisects the angle at $A$. In the same way, we can show that the other two angles at $C$ and $E$ are respectively bisected by $C Q$ and $E Q$. Since all six angle bisectors are concurrent at $Q$, it shows that $Q$ is equidistant from all six sides, and therefore completes the proof that an incircle exists.

## Alternative concurrency proof

In De Villiers \& Hung (2022) we proved that the main diagonals of the hexagon $A B C D E F$ are concurrent by using a theorem by Anghel (2016). However, since the hexagon has an incircle as
shown in the theorem above, the concurrency of the main diagonals $A D, B E$, and $C F$ follows immediately from the application of Brianchon's paper, and provides much easier proof.

It has also come to my attention that this hexagon concurrency result is apparently attributed to A . Zaslavsky, and a diagram (without proof) of it is given in Akopyan (2011, problem 4.9.26, p. 53). It also appeared earlier as a problem in the Third Sharygin Olympiad in Geometry (2007, Final Round, Grade 9, Problem 3). Though in Russian, it's easy to see that the given solution on p. 6 to Problem 3 of the Third Sharygin Olympiad Solutions (2007), is via Theorem 1 above (see p. 6, Fig. 9.3).

## Converse of Theorem 1

An equivalent formulation of Theorem 1 is the following: Given a hexagon $A B C D E F$ with $A B=B C, C D=D E, E F=F A$, and $\Varangle A=\Varangle C=\Varangle E$, then the angle bisectors of $\Varangle A, \Varangle C$, and $\Varangle E$ are concurrent at the circumcentre, $Q$, of $\triangle A C E$. This formulation now gives us the following neat converse: Given a hexagon $A B C D E F$ with $A B=B C, C D=$ $D E, E F=F A$, and the angle bisectors of $\Varangle A, \Varangle C$, and $\Varangle E$ are concurrent at the circumcentre, $Q$, of $\triangle A C E$, then $\Varangle A=\Varangle C=\Varangle E$.

Proof. Again consider Figure 1. It is given that $A Q$ and $C Q$ respectively bisect the angles at $A$ and $C$; thus $\Varangle B A F=2 \Varangle B A Q$ and $\Varangle D C B=2 \Varangle B C Q$. But as before $A B C Q$ is a kite. Therefore, $\Varangle B A Q=\Varangle B C Q$; thus $\Varangle B A F=\Varangle D C B$. Therefore, the two angles at $A$ and $C$ are equal. In the same way, we can show that the angle at $E$ is equal to either one of the angles at $A$ or $C$, to complete the proof that $\Varangle A=\Varangle C=\Varangle E$.

It's also interesting to explore when $A B C D E F$ is cyclic. A little exploring with the aid of a dynamic geometry sketch, quickly gave the following additional theorem.

## Theorem 2.

Given a hexagon $A B C D E F$ with $A B=B C, C D=D E, E F=F A$, and $\Varangle A=\Varangle C=\Varangle E$, then $A B C D E F$ is cyclic only when $\triangle A C E$ is equilateral, and the hexagon is regular. Proof. For $A B C D E F$ to be cyclic the points $B, D$ and $F$ have to lie on the circumcircle of $\triangle A C E$. Assume that $B$ lies on the circumcircle of $\triangle A C E$ as shown in Figure 2. Label $\Varangle B A C=x, \Varangle C A Q=$ $p, \Varangle E A Q=r$ and angle $F A E=z$. Then determine the other angles in the diagram through some straightforward angle chasing.
From Theorem 1, we have the following equation:

$$
\begin{equation*}
x+p=z+r \tag{1}
\end{equation*}
$$

Since $A B C E$ is a cyclic quadrilateral (by assumption/construction), $\Varangle A B C$ is supplementary to $\Varangle A E C$. Hence,

$$
\begin{equation*}
90^{\circ}-p=2 x \rightarrow 2 x+p=90^{\circ} \tag{2}
\end{equation*}
$$

Similarly, for $A C E F$ to be cyclic, $\Varangle A C E$ must be supplementary to $\Varangle A F E$. Hence,

$$
\begin{equation*}
90^{\circ}-r+180^{\circ}-2 z=180^{\circ} \rightarrow 2 z+r=90^{\circ} \tag{3}
\end{equation*}
$$

Equating Equations 2 and 2, gives $2 x+p=2 z+r$. Subtracting Equation 1 from the corresponding sides of the preceding equation, gives $x=z$. Substitution of $x=z$ back into Equation 1, also


Figure 14: Angles in hexagon
implies that $p=r$. Similarly, for $A C D E$ to be cyclic, $\Varangle C A E$ must be supplementary to $\Varangle C D E$. Hence, $360^{\circ}-2 p-4 r-2 z+p+r=180^{\circ} \rightarrow p+3 r+2 z=180^{\circ}$. But substituting $p=r$ from the above into this equation, gives

$$
\begin{equation*}
2 r+z=90^{\circ} \tag{4}
\end{equation*}
$$

Equating Equations 2 and 4 , gives $2 x+p=2 r+z$. Again subtracting Equation 1 from the corresponding sides of the preceding equation, gives $x=r$. Substituting $x=r$ and $p=r$ into Equation 1, gives $3 r=90^{\circ} \rightarrow r=30^{\circ}=x=p=z$. From the symmetry of the problem, it's obvious that the same relationships between the four angles at each of the vertices $C$ and $E$ would also hold. Therefore, if $A B C D E F$ is cyclic, $\triangle A C E$ will be equilateral, and the isosceles triangles on its sides, congruent to each other (with apex angles of $120^{\circ}$ ). Thus, $A B C D E F$ will be a regular hexagon.

## Alternative Construction

Theorem 1 and its converse provide an alternative, easier way to construct a dynamic version of $A B C D E F$ than the one implied by the results in De Villiers \& Hung (2022). From Theorem 1, one can easily construct $A B C D E F$ by starting with an arbitrary $\triangle A C E$ and its circumcentre, $Q$. Connect $Q$ with each the vertices $A, C$, and $E$. Choose an arbitrary point B on the perpendicular bisector of $A C$, and reflect line $A B$ around $A Q$. The point $F$ is then located at the intersection of the reflected line with the perpendicular bisector of $A E$. Repeat the same reflection with line
$F E$ around $E Q$ to locate point $D$ at the intersection of the reflected line with the perpendicular bisector of $C E$. The formed hexagon $A B C D E F$ can then be dynamically changed by dragging any of the vertices of $\triangle A C E$, or the variable point $B$.

## Further Generalization

It is not hard to see, and prove in the same way as before, that the converse of Theorem 1 generalizes as follows to an octagon: Given an octagon $A B C D E F G H$ with $A C E G$ cyclic, $A B=B C, C D=D E, E F=F G, G H=H A$, and the angle bisectors of $\Varangle A, \Varangle C, \Varangle E$ and $\Varangle G$ concurrent at the circumcentre, $Q$, of $A C E G$, then $\Varangle A=\Varangle C=\Varangle E=\Varangle G$ (see Figure 3). From the argument it's easy to see that the converse of Theorem 1 would further generalize in the same way to a decagon, and in general, to a $2 n$-gon with $n \geq 3$. Note that to construct a dynamic $2 n$-gon with this property, one can use the alternative construction described above. For example, for an octagon, one again starts with a cyclic quadrilateral $A C E G$, and an arbitrary point $B$, on the perpendicular bisector of $A C$, and then reflect line $A B$ around $A Q$, etc. Also note that since this construction produces a $2 n$-gon with all the angle bisectors concurrent at $Q$, it follows that $Q$ is equidistant from all the sides, and therefore the $2 n$-gon has an incircle.


Figure 15: Octagon generalization of converse of Theorem 1

Perhaps unexpectedly, Theorem 1 does not likewise generalize to an octagon. For example, Figure 4 provides a counter-example to the statement: Given an octagon $A B C D E F G H$ with
$A C E G$ cyclic, $A B=B C, C D=D E, E F=F G, G H=H A$, and $\Varangle A=\Varangle C=\Varangle E=\Varangle G$, then the angle bisectors of $\Varangle A, \Varangle C, \Varangle E$ and $\Varangle G$ are concurrent at the circumcentre, $Q$, of $A C E G$. The figure clearly shows that $\Varangle B A Q \neq \Varangle H A Q$, and therefore $A Q$ is not the angle bisector of $\Varangle A$.

However, Theorem 1 does generalize to a decagon as follows: Given a decagon $A B C D E F G H I J$ with $A C E G I$ cyclic, $A B=B C, C D=D E, E F=F G, G H=H I, I J=J A$, and $\Varangle A=\Varangle C=\Varangle E=\Varangle G=\Varangle$ then the angle bisectors of $\Varangle A, \Varangle C, \Varangle E, \Varangle G$ and $\Varangle I$ are concurrent at the circumcentre, $Q$, of $A C E G$ (see Figure 5).

With this arrangement of the kites and the equal angles at vertices $A, C, E, G, I$, the same proof of Theorem 1 can again be used and is left to the reader to complete. Note that Theorem 1 can therefore be generalized to a $2 n$-gon where $n$ is odd and $n \geq 3$. In addition, since all the angle bisectors are again concurrent at $Q$, these $2 n$-gons will all have incircles.


Figure 16: Octagon counter-example for generalization of Theorem 1

## Concluding Remarks

The proof of Theorem 1 needs some modification for the cases when the circumcentre, $Q$, of $\triangle A C E$ lies outside the triangle. However, these modifications can be avoided by stating and consistently using directed angles through-out. Further reflection and investigation of the hexagon


Figure 17: Decagon generalization of Theorem 1
in question not only produced some other interesting properties, but also a simpler proof of the concurrency of the main diagonals, as well as some generalizations to $2 n$-gons. This demonstrates the value of 'looking back' as advocated by Polyá (1945). Over-all, the problems are relatively straight forward and quite suited for use in a problem solving course with novice learners and students or for some basic practice for a mathematics competition at an introductory level.
Web Supplement.
http://dynamicmathematicslearning.com/further-hexagon-propertie s.html

Disclaimer. No potential competing interest was reported by the authors.

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Michael de Villiers
Mathematics Education
University of Stellenbosch, South Africa
Dynamic Geometry Sketches:

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http://dynamic-mathematicslearning.com/JavaGSPLinks.htm
profmd1@mweb.co.za
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