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A generalized dual of Napoleon's theorem and some further extensions

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Utilizing a duality between the concepts incentre and circumcentre, a dual to a well-known generalization of Napoleon's theorem is conjectured, experimentally confirmed and eventually proved. The proof then also shows that the result is merely a special case of a more general result. As a further consequence, two interesting related results are also derived.

A useful duality is that between the concepts *angle bisector* and *perpendicular bisector*. For instance, an angle bisector and perpendicular bisector are both loci of points, respectively equidistant from two lines intersecting in a point (an angle) or two points lying on a line (a line segment). Likewise an *incentre* and *circumcentre* are each other's duals, both being points, respectively equidistant from a number of line segments and a number of points (vertices).

Although it is not a general duality like the duality between points and lines in projective geometry, theorems involving these concepts frequently also occur in dual pairs. The following examples are from De Villiers [1]:

(1) The angle bisectors of any circum polygon (a polygon circumscribed around a circle) are concurrent at the incentre of the polygon.
(2) The perpendicular bisectors of any cyclic polygon are concurrent at the circumcentre of the polygon.
(3) If O is the incentre of any triangle ABC, then the circumcentre of triangle BOC, say P, lies on the angle bisector of angle A, namely $\overrightarrow{AO}$ (Figure 1).
(4) If O is the circumcentre of any triangle ABC, then the incentre of triangle BOC, say P, lies on the perpendicular bisector of BC (the side opposite angle A), namely $\overrightarrow{EO}$, where E is the midpoint of BC (Figure 2).
(5) Steiner Lehmus: any triangle that has two equal angle bisectors (each measured from the vertex to the opposite side) is isosceles (see Figure 3).
(6) Any triangle that has two equal perpendicular bisectors (each measured from the midpoint to the opposite side) is isosceles (see Figure 4).

Since the incentre and circumcentre of an equilateral triangle coincide, Napoleon's

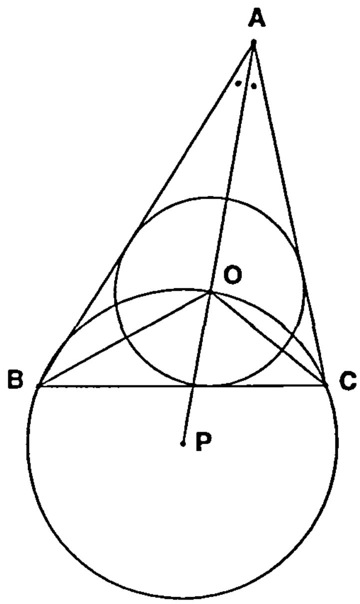

Figure 1.

theorem is *selfdual* with respect to these concepts, since it states that if equilateral triangles are erected (outwardly or inwardly) on the sides of any triangle, their circumcentres form an equilateral triangle (see Wetzel [2]).

A well-known generalization of Napoleon's theorem is the following:

> If similar triangles ADB, CBE and FAC are erected on the sides of any triangle ABC, their circumcentres, G, H and I form a triangle similar to the three triangles (see Coxeter and Greitzer [3]).

The preceding duality now suggests the following dual which was confirmed experimentally on a computer, using *Cabri-Géomètre* (see De Villiers [4]):

> If similar triangles ADB, CBE and FAC are erected outwardly on the sides of any triangle ABC, their incentres G, H and I form a triangle similar to the three triangles.

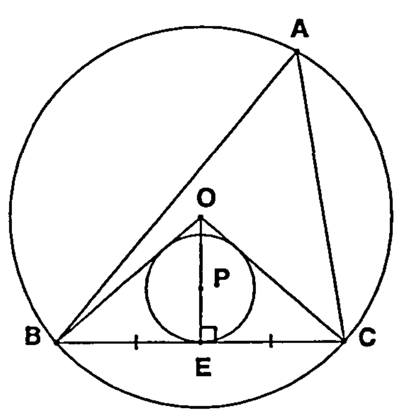

Figure 2.

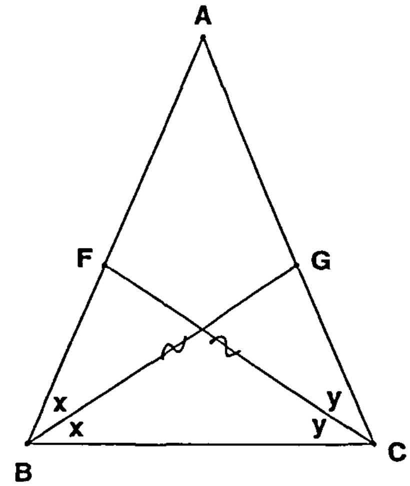

Figure 3.

Proof. This result can be proved using mainly the following special case of the Petersen–Schoute theorem (Coxeter and Greitzer [3]):

> If ABC and $A'B'C'$ are two directly similar triangles, while $AA'A''$, $BB'B''$, $CC'C''$ are three directly similar triangles, then $\triangle A''B''C''$ is directly similar to $\triangle ABC$ (Figure 5).

Let us now first examine what we mean by points that are in the same relative position with respect to two directly similar triangles. (Two triangles are directly similar if the transformation which maps the one triangle on to the other preserves angles in both magnitude and direction.)

Assume two triangles ABC and $A'B'C'$ are directly similar (Figure 6). Let P be any point in the plane. Then we can say that the point P' is in the same relative position to $\triangle A'B'C'$ as P is to $\triangle ABC$ if the transformation (in this case a translation or a spiral similarity) which maps $\triangle ABC$ to $\triangle A'B'C'$, also maps the point P onto P'.

Let us now consider the above generalized dual to Napoleon's theorem. In Figure 7 we have $\triangle ABC$ and the three directly similar triangles ADB, CBE and FAC. Now

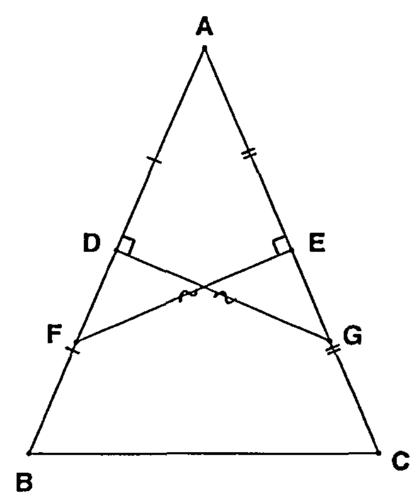

Figure 4.

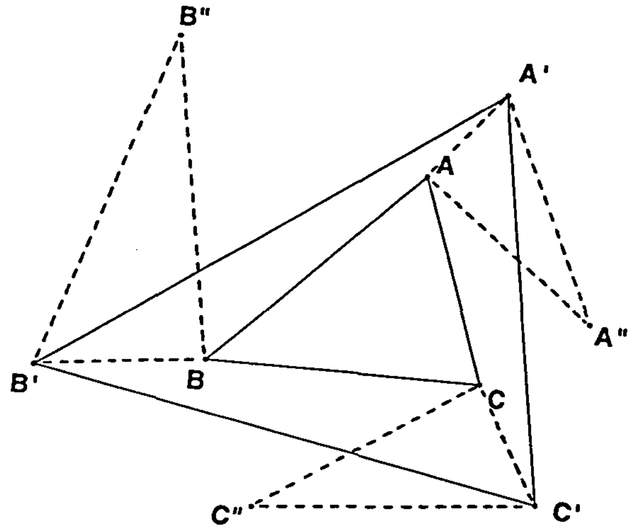

Figure 5.

choose any three points P, Q and R in the plane so that they are, respectively, in the same relative positions to triangles ADB, CBE and FAC. (We could for example choose the three incentres.) Let G, H and I be the three circumcentres of the similar triangles.

Then, according to the aforementioned generalization of Napoleon's theorem, we have ΔGHI directly similar to the three similar triangles. We shall now use the Petersen–Schoute theorem to show that ΔPQR is directly similar to ΔGHI which then provides the desired result.

First, we see that triangles PAG, QCH and RFI are directly similar since the points P, A and G have relatively the same positions to ΔADB as the points Q, C and H, respectively, have to ΔCBE. The points R, F and I similarly have the same relative positions to ΔFAC as the points Q, C and H have to ΔCBE. In Figure 8 we therefore have the same configuration as that of Figure 5 and the result now follows directly from the Petersen–Schoute theorem. (Note that triangle $A'B'C'$ in

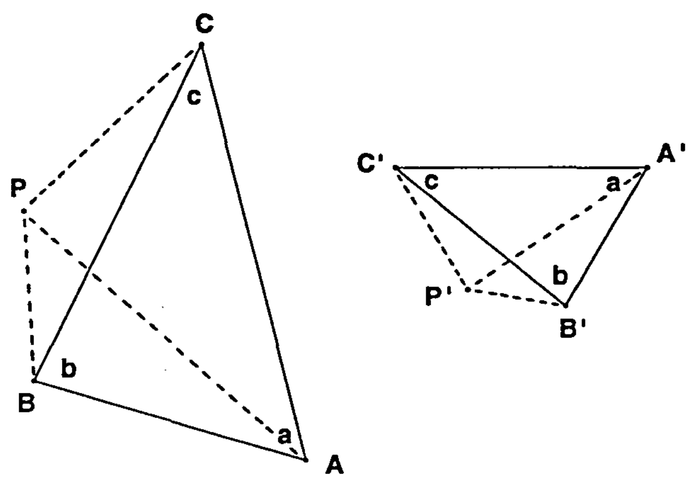

Figure 6.

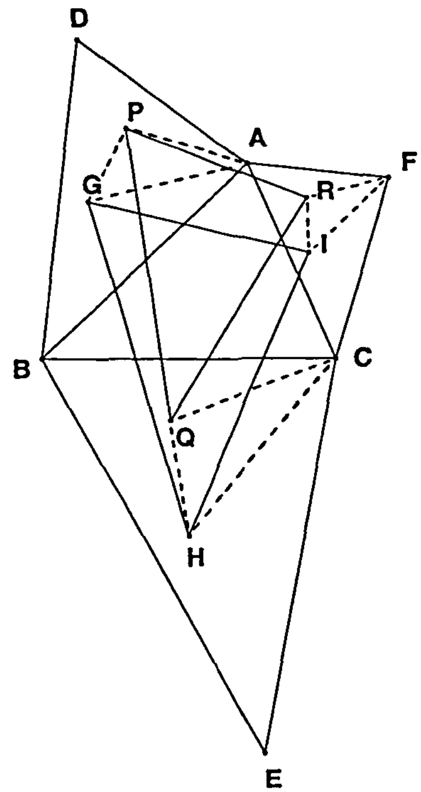

Figure 7.

Figure 5 does not have to include triangle ABC.) It should also be observed that $Q\hat{P}R = B\hat{D}A$, $P\hat{Q}R = B\hat{E}C$ and $Q\hat{R}P = C\hat{F}A$.

The aforementioned generalization of Napoleon's theorem, as well as its dual, is therefore merely a special case of the following more general result:

Theorem 1. If similar triangles ADB, CBE and FAC are erected outwardly on the sides of any triangle ABC, and any three points P, Q and R are chosen so that

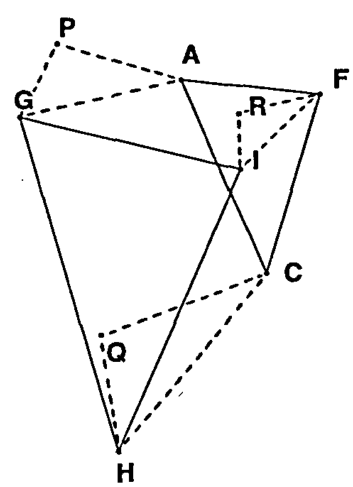

Figure 8.

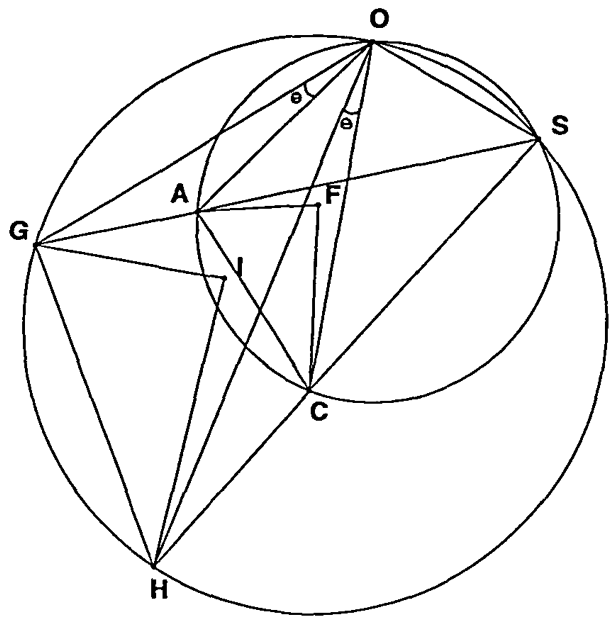

Figure 9.

they respectively lie in the same relative positions to these triangles, then P, Q and R form a triangle similar to the three triangles.

An interesting corollary is that we would also obtain a similar triangle if we instead connected the corresponding centroids or orthocentres of the similar triangles.

Note. It is interesting to examine how the point O around which the rotation takes place should be constructed (when the triangles GHI and ACF are not congruent and cannot be mapped onto each other by a translation, in which case the whole situation is trivial). Consider for example in Figure 9 the line segments GH and AC which are mapped onto each other by means of a similar transformation. Let GA and HC (extended) meet in S and let the other intersection of the circles through ACS and GHS be O. (If the circles touch each other at S, choose O to be the same point as S). The point O is the desired point, since $O\hat{H}G = O\hat{S}G = O\hat{S}A = O\hat{C}A$ and similarly $O\hat{A}C = O\hat{G}H$. This means that triangles OGH and OAC are directly similar by means of the spiral similarity $O(\kappa, \theta)$ where $\kappa = AC/GH$ and $\theta = G\hat{O}A$. Since $H\hat{G}I = C\hat{A}F$ and $G\hat{H}I = A\hat{C}F$, the point I would necessarily map onto F under this transformation, in other words triangles ACF and GHI are directly similar under the transformation $O(\kappa, \theta)$. This result has a useful application in our proof of Theorem 2 later on.

Another less well-known generalization of Napoleon's theorem is to alter the arrangement of the triangles on the sides by outwardly constructing three similar triangles ABD, EBC and AFC, instead of the similar triangles ADB, CBE and FAC. The circumcentres G, H and I of these triangles also form a triangle similar to them, since we still have $\hat{D} + \hat{E} + \hat{F} = 180°$, which is the same characteristic property upon which the more familiar formulation depends [3].

Using *Cabri Géomètre* again in checking the corresponding case for incentres for

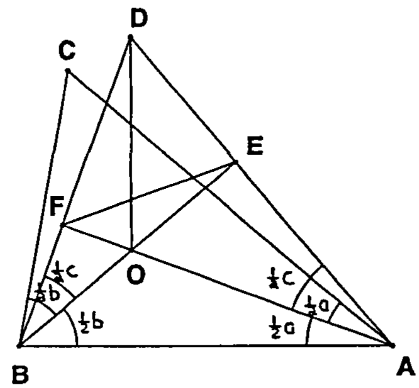

Figure 10.

this arrangement of triangles, it was found that triangle GHI was no longer similar to triangles ABD, EBC and AFC. However, this investigation has led to the following discovery [4].

Theorem 2. If similar triangles ABD, EBC and AFC are erected outwardly on the sides of any triangle ABC, their incentres G, H and I form a triangle with $\hat{G} = \frac{1}{2}(D\hat{A}B + D\hat{B}A)$, $\hat{H} = \frac{1}{2}(E\hat{B}C + E\hat{C}B)$ and $\hat{I} = \frac{1}{2}(F\hat{C}A + F\hat{A}C)$ (Figure 11).

Let us now first prove the following lemma:

Lemma 1. Consider a triangle ABC with angles a, b and c and incenter O as indicated in Figure 10. If we construct a new triangle DBA with $D\hat{B}O = \frac{1}{2}c$ and $D\hat{A}O = \frac{1}{2}c$, then $B\hat{D}O = \frac{1}{2}a$ and $A\hat{D}O = \frac{1}{2}b$.

Proof. Extend BO and AO to E and F on AD and BD, respectively. Then $EFBA$ is a cyclic quadrilateral, since $F\hat{B}E = F\hat{A}E = \frac{1}{2}c$. Therefore $E\hat{F}A = E\hat{B}A = \frac{1}{2}b$. But $OEDF$ is also a cyclic quadrilateral, since $F\hat{O}E + F\hat{D}E = 180°$. Therefore $O\hat{D}E = O\hat{F}E = \frac{1}{2}b$. Since $B\hat{D}A = \frac{1}{2}(a+b)$ it follows that $O\hat{D}B = \frac{1}{2}a$.

Proof of Theorem 2. In Figure 11 we have triangle ABC with three directly similar triangles ABD, EBC and AFC with angles a, b and c as indicated. Let G, H and I indicate the respective incentres.

Construct triangles ABP, CQB and RCA in the same way as in Lemma 1, in other words enlarge $G\hat{A}B$ and $G\hat{B}A$ each by $\frac{1}{2}c$ to obtain the point P, enlarge $H\hat{B}C$ and $H\hat{C}B$ each by $\frac{1}{2}a$ to obtain Q and enlarge $I\hat{A}C$ and $I\hat{C}A$ each by $\frac{1}{2}b$ to obtain R. These three new triangles now stand on the sides of triangle ABC in the same way as in Theorem 1, with the angles P, Q and R, respectively, equal to $\frac{1}{2}(a+b)$, $\frac{1}{2}(b+c)$ and $\frac{1}{2}(a+c)$.

In addition, according to Lemma 1 we have that point G lies in the same relative position to $\triangle ABP$, as the point H lies relatively to $\triangle CQB$. The same is true for the point I relative to $\triangle RCA$. According to Theorem 1, we therefore obtain the desired result that $\hat{G} = \frac{1}{2}(a+b)$, $\hat{H} = \frac{1}{2}(b+c)$ and $\hat{I} = \frac{1}{2}(a+c)$.

We can now also prove the following interesting theorem:

Theorem 3. If triangles DBA, ECB ans FAC with angles α, β and γ are erected outwardly on the sides of any triangle ABC as shown in Figure 12, and $\alpha + \beta + \gamma = 90°$, then $E\hat{D}F = 2\beta$, $D\hat{F}E = 2\alpha$ and $F\hat{E}D = 2\gamma$.

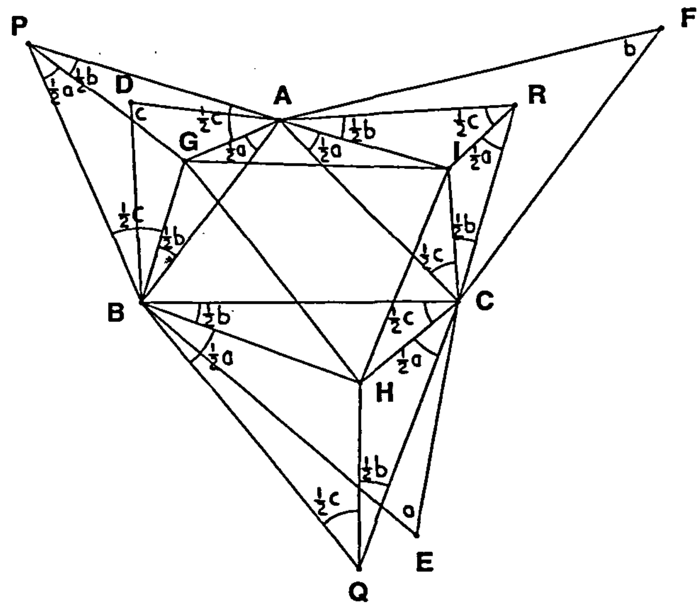

Figure 11.

Proof. Construct three new triangles PBA, BQC and ACR by choosing $P\hat{B}D=\gamma$, $P\hat{A}D=\alpha$, $Q\hat{B}E=\beta$, $Q\hat{C}E=\alpha$, $R\hat{C}F=\gamma$ and $R\hat{A}F=\beta$. Then triangles PBA, BQC and ACR are directly similar with incentres D, E and F, respectively. Therefore ΔDEF is directly similar to ΔPBA so that $E\hat{D}F=B\hat{P}A=2\beta$, etc.

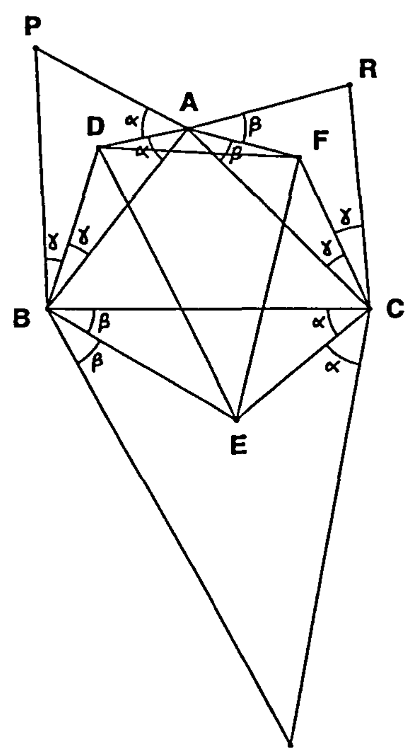

Figure 12.

References

[1] DE VILLIERS, M. D., 1992, *Math. Digest*, **89,** 4.
[2] WETZEL, J. E., 1992, *Amer. Math. Monthly*, **99,** 339.
[3] COXETER, H. S. M., and GREITZER, S. L., 1967, *Geometry Revisited*, New Mathematical Library, Vol. 19 (Washington, DC: MAA) pp. 61–63; 99.
[4] DE VILLIERS, M. D., 1991, *Pythagoras*, **27,** 43.