

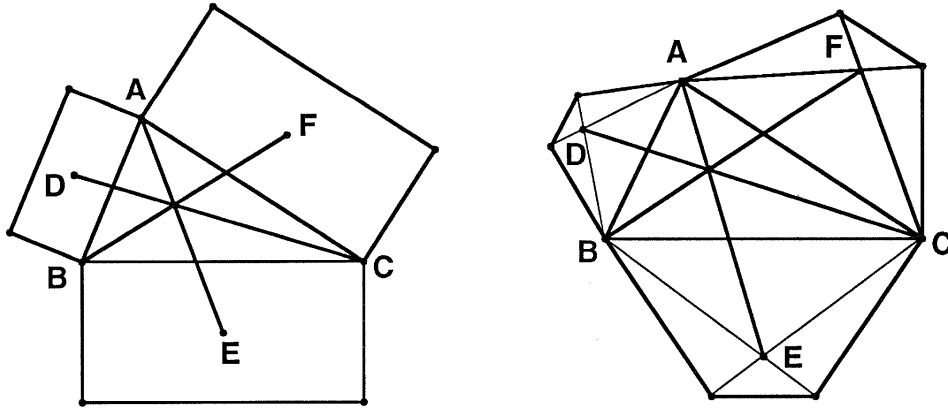


Figure 4.53

17. In an article in the **Mathematical Digest** Murray Klamkin (1991) pointed out that an analogous generalization exists for cyclic star $4n$ -gons, namely:
- (1) If $A_1A_2\dots A_{4n}$ ($n>1$) is any cyclic star polygon P in which each vertex A_i is joined to vertex A_{i+2n-1} , then the two sums of alternate interior angles of P are each equal to π (see Figure 4.54a where $n = 2$).

However, as before, a dual also exists for this generalization (compare De Villiers, 1991):

- (2) If $A_1A_2\dots A_{4n}$ ($n>1$) is a circumscribed star polygon P in which each vertex A_i is joined to vertex A_{i+2n-1} , then the two sums of alternate sides are equal (see Figure 4.54b where $n = 2$).

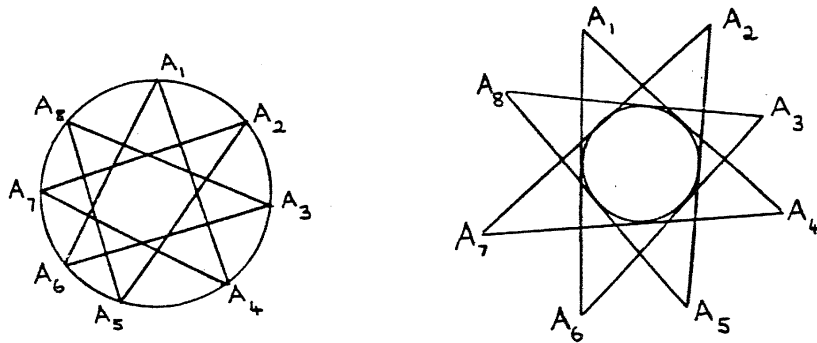


Figure 4.54

A unifying generalization

It is furthermore possible as shown in De Villiers (1993) to construct a class of cyclic $2n$ -gons ($n>1$) with the connecting rule $A_i \rightarrow A_{i+n-1}$ as shown in Figure 4.55 for which the total interior angle sum is 2π and the two sums of alternate angles are therefore equal to π . (Note that the cases $n=4, 6, 8, 10\dots$ are the same polygons referred to in (1) above). The cases for $n=3, 5, 7\dots$ can also be seen to consist of a set of two figures, i.e.

triangles (a generalized star of David), star pentagons, star septagons, etc., respectively overlapping in such a manner that $A_1; A_3; A_5 \dots A_{2n-1}$ belong to the one figure and $A_2; A_4; A_6 \dots A_{2n}$ belong to the other figure. (Note therefore that in these cases the polygons need not be cyclic for the result to be true).

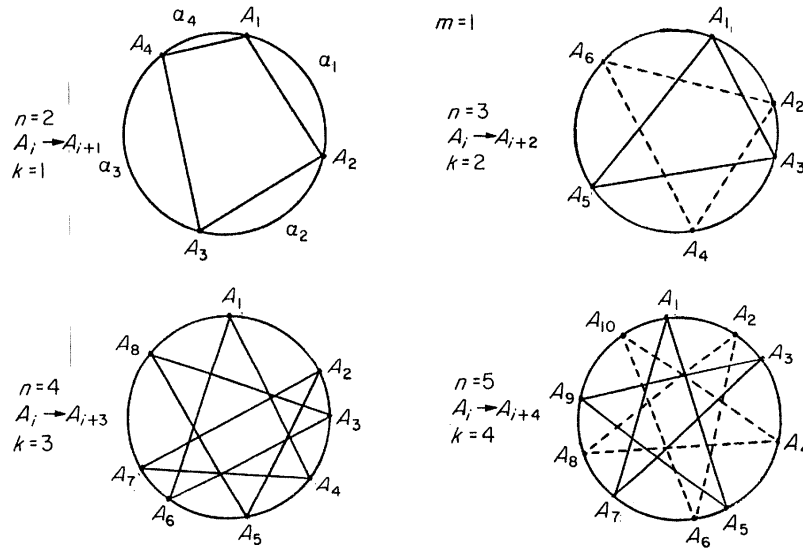


Figure 4.55

Next we can construct a class of cyclic $2n$ -gons ($n > 2$) with the connecting rule $A_i \rightarrow A_{i+n-2}$ as shown in Figure 4.56, for which the total interior angle sum is 4π and the two sums of alternate interior angles are therefore equal to 2π . Here the cases for $n=4$ and $n=6$ can also be seen to consist respectively of two overlapping quadrilaterals and two generalized stars of David, and therefore need not be cyclic.

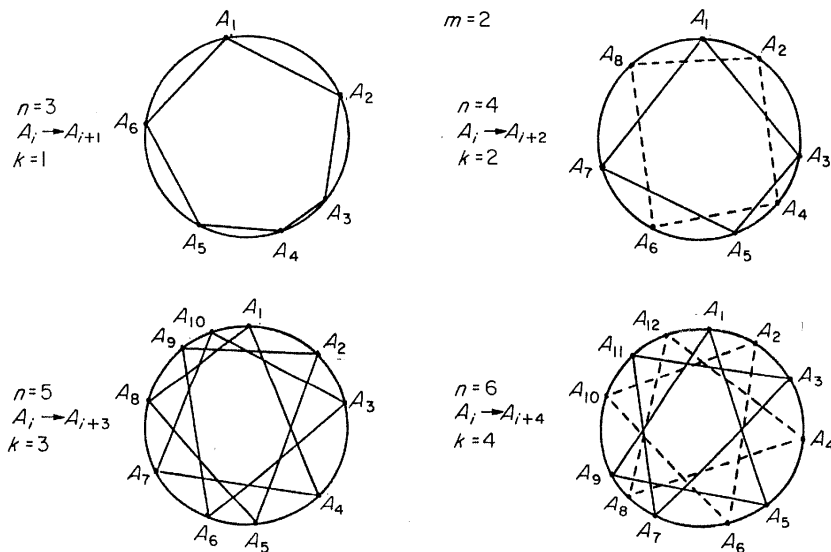


Figure 4.56

In a similar fashion other general classes can also be formed from convex cyclic octagons, decagons, etc. The connecting rule would then in general simply be $A_i \rightarrow A_{i+n-m}$ with $m = 1, 2, 3, \dots$ and $n - m \geq 1$. Or alternatively and more simply, the general connecting rule can be formulated as $A_i \rightarrow A_{i+k}$ where $k = 1, 2, 3, \dots$ is the **total turning** (the number of complete rotations of 2π) one would undergo walking completely around the perimeter of each figure. We can therefore now formulate the following beautiful generalization of Turnbull's theorem which includes all the previous cases:

Theorem 1

If $A_1A_2\dots A_{2n}$ ($n > 1$) is any cyclic $2n$ -gon in which vertex $A_i \rightarrow A_{i+k}$ (vertex A_i is joined to A_{i+k}), then the two sums of alternate interior angles are each equal to $m\pi$ (where $m = n - k$).

Although generalization (1) given earlier can easily be proved by mathematical induction from the special case for cyclic quadrilaterals and considering the addition of two vertices at a time, the following proof, based on the notation and approach of Klamkin (1991), turned out more convenient for the unifying generalization.

Proof

Let the measure of the minor arc $A_iA_{i+1} = \alpha_i$; $i = 1; 2; \dots; 2n$ with $\alpha_{2n+1} = \alpha_1$. (See first figure in Figure 4.55). First consider the case for $m = 1$ and the sum of the odd alternate angles, then:

$$\begin{aligned} 2\angle A_1 &= \alpha_n + \alpha_{n+1} \\ 2\angle A_3 &= \alpha_{n+2} + \alpha_{n+3} \\ &\vdots \\ 2\angle A_{2n-1} &= \alpha_{n-2} + \alpha_{n-1} \end{aligned}$$

Then by addition we have: $2\sum_{i=1}^n \angle A_{2i-1} = 2\pi \Leftrightarrow \sum_{i=1}^n \angle A_{2i-1} = \pi$.

In the same manner we can prove that the sum of the even alternate interior angles is also π . (Or alternatively we can in general find the sum of the even alternate angles simply by subtracting πm from the total interior angle sum of these $2n$ -gons, namely $S = 2\pi m$. This follows directly as a special case from the general proof for n -gons given in *Solutions 3*, no. 15 (compare De Villiers, 1989)).

After also considering the case for $m = 2$, the general argument is easy to formulate as follows:

$$\begin{aligned} 2\angle A_1 &= \alpha_{n-m+1} + \alpha_{n-m+2} + \dots + \alpha_{n+m-1} + \alpha_{n+m} \\ 2\angle A_3 &= \alpha_{n-m+3} + \alpha_{n-m+4} + \dots + \alpha_{n+m+1} + \alpha_{n+m+2} \\ &\vdots \\ 2\angle A_{2n-1} &= \alpha_{n-m-1} + \alpha_{n-m} + \dots + \alpha_{n+m-3} + \alpha_{n+m-2} \end{aligned}$$

Then by addition and rearrangement we have:

$$\begin{aligned} 2\sum_{i=1}^n \angle A_{2i-1} &= m(\alpha_{n+m} + \alpha_{n+m+1} + \dots + \alpha_{n+m-2} + \alpha_{n+m-1}) = m(2\pi) \\ \Leftrightarrow \sum_{i=1}^n \angle A_{2i-1} &= m\pi \end{aligned}$$

Or alternatively, mathematical induction can be used in an informative manner as follows. As it is true for $m = 1$ as already shown, let's assume that it is true for $m = p$, and therefore that the following is true:

$$\begin{aligned} 2\angle A_1 &= \alpha_{n-p+1} + \alpha_{n-p+2} + \dots + \alpha_{n+p-1} + \alpha_{n+p} \\ 2\angle A_3 &= \alpha_{n-p+3} + \alpha_{n-p+4} + \dots + \alpha_{n+p+1} + \alpha_{n+p+2} \\ &\vdots \\ 2\angle A_{2n-1} &= \alpha_{n-p-1} + \alpha_{n-p} + \dots + \alpha_{n+p-3} + \alpha_{n+p-2} \\ \Rightarrow 2\sum_{i=1}^n \angle A_{2i-1} &= p(2\pi) \Leftrightarrow \sum_{i=1}^n \angle A_{2i-1} = p\pi. \end{aligned}$$

Now consider $m = p + 1$, then:

$$\begin{aligned} 2\angle A_1 &= \alpha_{n-p} + \alpha_{n-p+1} + \dots + \alpha_{n+p} + \alpha_{n+p+1} \\ 2\angle A_3 &= \alpha_{n-p+2} + \alpha_{n-p+3} + \dots + \alpha_{n+p+2} + \alpha_{n+p+3} \\ &\vdots \\ 2\angle A_{2n-1} &= \alpha_{n-p-2} + \alpha_{n-p-1} + \dots + \alpha_{n+p-2} + \alpha_{n+p-1} \end{aligned}$$

By addition, rearrangement and utilization of the assumed truth for $m = p$ we then obtain:

$$2\sum_{i=1}^n \angle A_{2i-1} = p(2\pi) + 2\pi \Leftrightarrow \sum_{i=1}^n \angle A_{2i-1} = (p+1)\pi.$$

This shows that it is true for $m = p + 1$, but the argument is true for $m = 1$, and therefore according to the principle of mathematical induction, it would be true for all $m = 1, 2, 3, \dots$

Furthermore, in accordance with generalizations (1) and (2) above, the above unifying generalization of Turnbull's theorem has the following interesting dual:

Theorem 2

If $A_1A_2\dots A_{2n}$ ($n > 1$) is any circumscribed $2n$ -gon in which vertex $A_i \rightarrow A_{i+k}$, then the two sums of alternate sides are equal.

In Figure 4.57, the first four dual examples are given for $m=1$. Note that in theorem 1 the sum of the alternate angles is equal to half the total angle sum, while in the dual result we have the sum of the alternate sides equal to half the perimeter. However, in contrast to Theorem 1, the two sums of alternate sides are not constant for the same value of m , as the perimeter of a circumscribed $2n$ -gon varies, not only with respect to the size of the circle, but also for a circle with a fixed radius. Furthermore, it should be noted that it is necessary here to give a more general meaning to the concept

"side". For example, for $n = 3$ the "sides" are to be interpreted respectively as $A_1B_1 + A_2B_6$, etc. In other words, a "side" is the sum of the intersecting tangents drawn from adjacent vertices. As before, the proof of Theorem 2 is based on the theorem that the tangents drawn from a point outside a circle are equal, and is left to the reader.

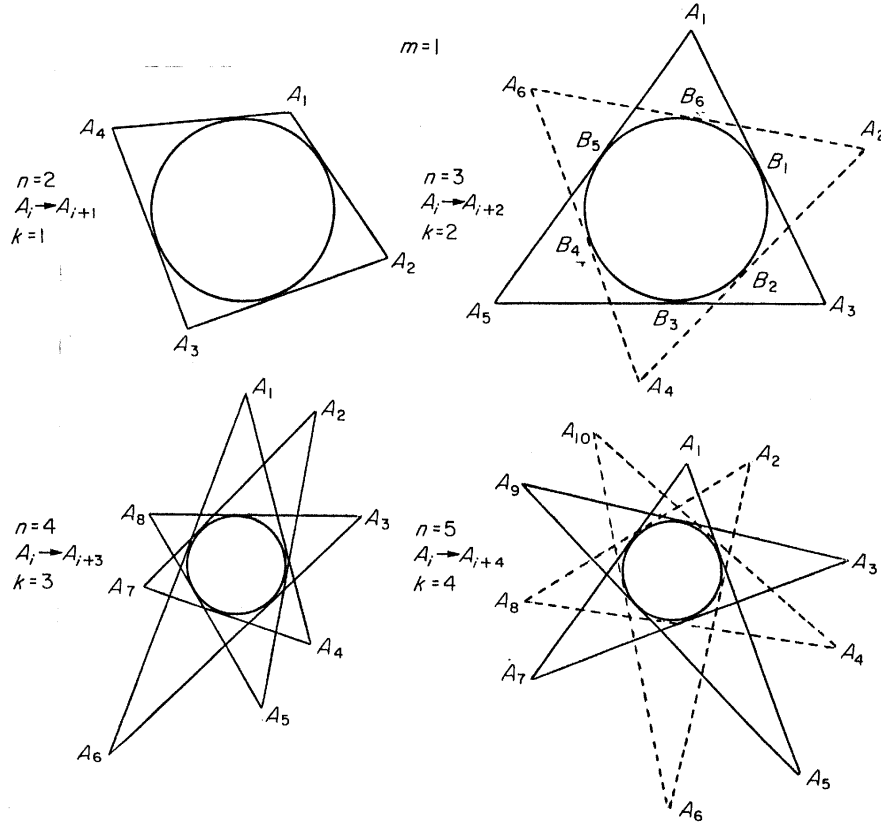


Figure 4.57

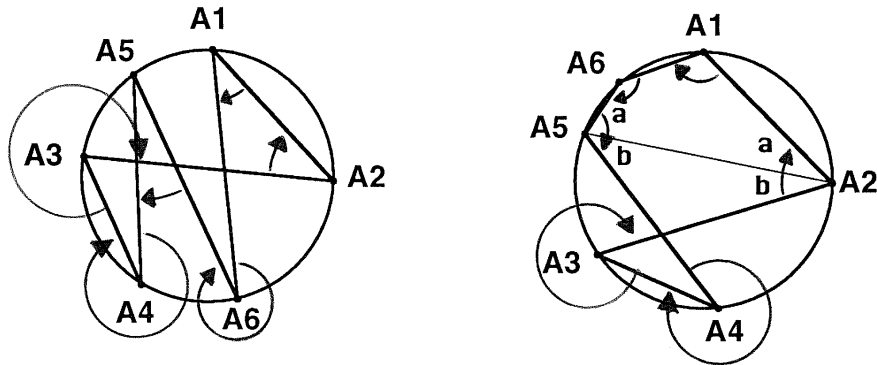


Figure 4.58

Further extensions?

What happens if we have crossed cyclic hexagons as shown in Figure 4.58? In the first case, we do not have the sums of alternate angles equal since $\angle A_1 + \angle A_3 + \angle A_5 = 2\pi$ and $\angle A_2 + \angle A_4 + \angle A_6 = 4\pi$ (the proof is left to the reader). However, in the second case the result holds.

For example, draw A_2A_5 then we have $\angle A_1 + \angle A_{5a} = \angle A_{2a} + \angle A_6 = \pi$ and $\angle A_{5b} + \angle A_3 = \angle A_{2b} + \angle A_4 = 2\pi$. By addition we therefore obtain the desired result, namely: $\angle A_1 + \angle A_3 + \angle A_5 = \angle A_2 + \angle A_4 + \angle A_6 = 3\pi$.

Can you further generalize this result? Is this result included in the previous generalization or can the generalization be reformulated to include it? Can you formulate a dual result? Is the dual result true or not? Investigate.

18. (b) It is easy to see that the similarity of triangles AOE and COF are maintained if we construct similar rectangles or rhombi as shown in Figure 4.59a-b. In fact, we can easily generalize it further to similar isosceles trapezia or kites as shown in Figure 4.59c-d. Can you generalize it still further?

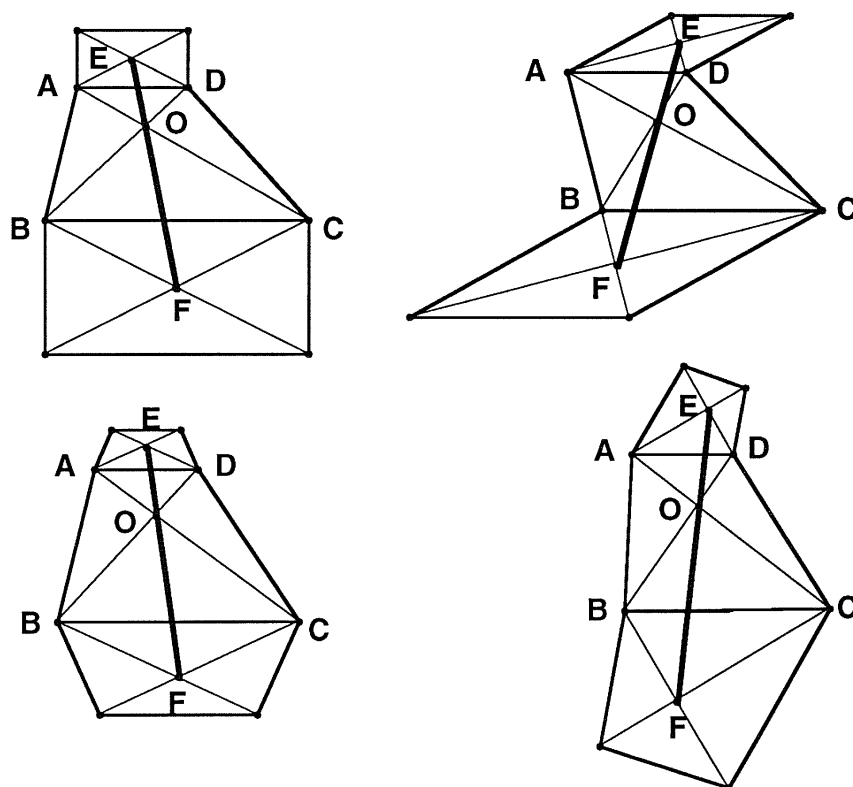


Figure 59

20. (c) This very beautiful result is known as Von Aubel's theorem and is true for convex, concave or crossed quadrilaterals and the squares can be constructed outward or inward. A general vector proof is given in Kelly (1966).

Note that the line segments joining the centres of opposite squares (the diagonals of EFGH) need not intersect, but nevertheless remain equal and perpendicular. The result can be *specialized* in a number of different ways, for example: