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Over and over again: two geometric iterations with triangles

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"*One can only reach infinity … by taking one step at a time.*"

Iteration in mathematics, doing the same procedure over and over, with each successive step using the results of the previous step, is a very fundamental concept and procedure in mathematics dating back to ancient times.

The famous method of Hero (who lived in Alexandria, North Africa round about 100 BC) for finding the square root of a number *r* is a classic example. One starts the process by guessing a square root *x*, and then one divides the number *r* by one's guess *x*. Then taking the average of the original and the quotient of the division, gives one the next guess. One then repeats and continues the process to find the square root of *r* to whatever level of accuracy is required. In modern notation, Hero's method can be very simply expressed in the following recursive formula: $x_{n+1} = (x_n + r / x_n)/2$.

Though doing repetitive calculations or constructions like these are tedious if done by hand, this has completely changed with the advent of the computational age of electronic calculators and computers. Apart from specialized mathematical software like *Mathematica*, one can now use easily use ordinary scientific calculators, programmable graphing calculators and/or spreadsheets to investigate many different kinds of iterations at school level.

Iterative numerical methods like these are indispensable for mathematicians, scientists and engineers as many problems within mathematics, as well as in the application of mathematics to the real world cannot be solved algebraically, and can only be solved numerically. In general, for example, polynomial equations of order 5 and higher cannot be solved algebraically, and the proof of this by Abel in the 19th century is one of the great triumphs of abstract algebra.

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My third and fourth year students in Differential Equations are completely baffled every year when early in the course I write the following simple equation on the board and ask them to solve it for *x*: $x^2 = 2^x$. They usually start by taking logarithms both sides, but then immediately get stuck, and have no idea how to proceed further. Only after letting them stew for a little while, would I suggest that they try 'trial-and-error', the most basic numerical method, to find the two positive roots, and then usually by computer graphing, the negative root. Of course, this early activity is intended as an '*advance organizer*' (Ausubel et al, 1978) for later introducing my students to numerical methods for solving differential equations that cannot be solved algebraically.

Though I've become accustomed to my students' inability to solve this simple equation, their slavish adherence to algebraic manipulation as the only method to solve equations is a deeply worrying aspect of their general mathematical education. One can similarly ask to what extent, if any, are our learners at school currently exposed to numerical methods of solving equations, and to what extent do many of them as a consequence of the lack of attention, develop the misconception that all polynomial equations can be solved algebraically, Indeed, this misconception was found among the vast majority of prospective mathematics teachers in their final year at South African universities in 1983 (De Villiers, 1987).

Iteration is, of course, not restricted to algebra, calculus and number theory (e.g. sequences and series), but is a cornerstone of the modern fields of chaos and fractals. The purpose of this paper is to present two iterative construction procedures within Euclidean geometry that lead to interesting conjectures, as well as their proof, which ought to be within range of high school learners and their teachers.

The first problem I saw perhaps 15 years or so ago, I think, in an issue of the *Mathematical Gazette*, The second one, I only recently discovered experimentally using the dynamic geometry software, *Sketchpad*, which allows one to save any set of constructions as a tool and to quickly re-apply it iteratively. Or alternatively, one could use the 'Iterate' or 'Define Custom Transform' tools of *Sketchpad* for the investigations below.

The reader is now also invited to go online and dynamically experience the following two investigations by going to the following ready-made *JavaSketchpad* sketches at: http://dynamicmathematicslearning.com/equilateral-convergence.html

Investigation 1: Tangent points of incircle

Start with any ΔABC and its incircle as shown in Figure 1. Label the points where the circle touches the sides BC, CA, and AB respectively as A_1, B_1 and C_1. Repeat the process with the new $\Delta A_1B_1C_1$. Then again, and again, etc. What do you visually notice about the shape, and the displayed values of the angles of $\Delta A_nB_nC_n$ as n increases? Check by dragging vertices A, B or C. Can you make a conjecture? Can you explain why (prove that) your conjecture is true?

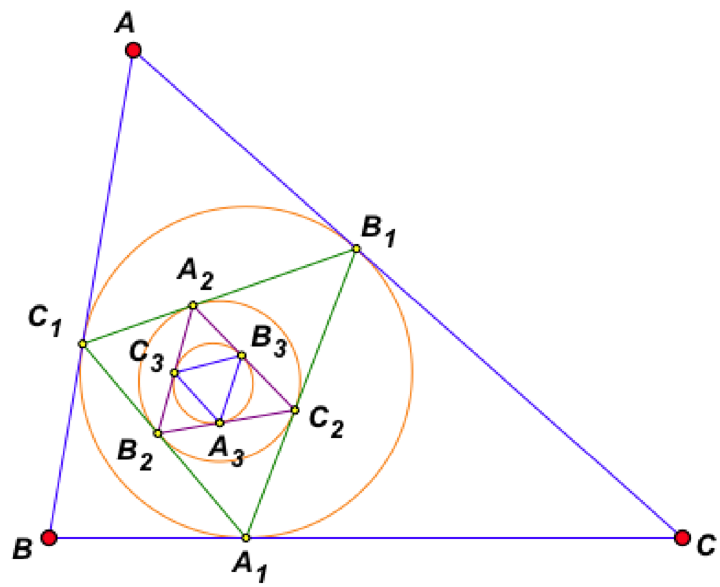

Figure 1: Iteration of tangent points of incircle

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Investigation 2: Excentres

Start with any ΔABC and construct its excentres[1]. Label the excentres formed on the sides of the sides BC, CA, and AB respectively as A_1, B_1 and C_1. Repeat the process with the new $\Delta A_1B_1C_1$. Then again, and again, etc. What do you visually notice about the shape, and the displayed values of the angles of $\Delta A_nB_nC_n$ as n increases? Check by dragging vertices A, B or C. Can you make a conjecture? Can you explain why (prove that) your conjecture is true?

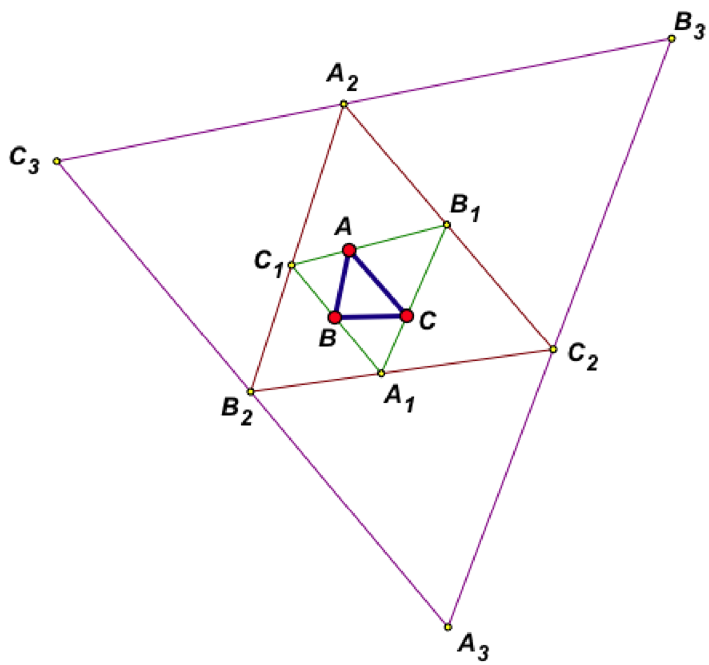


Figure 2: Iteration of excentres

In both cases, it is pleasingly interesting to visually note that at each successive stage $\Delta A_nB_nC_n$ becomes closer and closer to appearing to be equilateral, and it is natural to conjecture that as n increases to infinity, that $\Delta A_nB_nC_n$ converges towards an equilateral triangle.

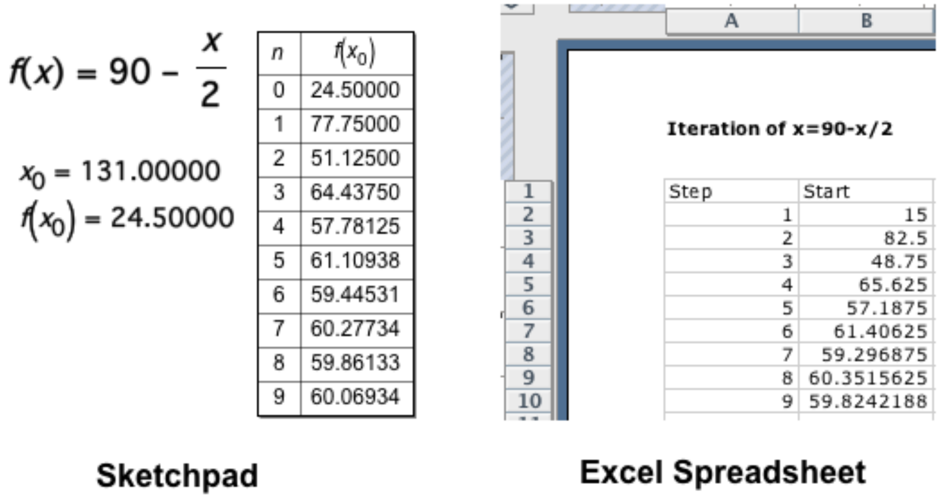
[1] The three excentres of a triangle are located at the intersection of the angle bisectors of the two exterior angles formed on each side of the triangle. For more information, go to: http://en.wikipedia.org/wiki/Incircle_and_excircles_of_a_triangle

Proofs

We shall first deal with Conjecture 1. If we let $\angle A = x$, then $\angle AC_1B_1 = 90^\circ - x/2$, since tangents from A to the circle are equal. From the tan-chord theorem, it follows that $\angle C_1A_1B_1 = 90^\circ - x/2$. Since the angle at A_2 is determined by the angle at A_1 in exactly the same way, we have the following recursive relationship for the successive values of $\angle A_n$: $x_{n+1} = 90^\circ - \frac{x_n}{2}$. Solving the equation $x = 90^\circ - \frac{x}{2}$ produces $x = 60^\circ$, which is clearly a unique *fixed point*[2]. The derivative of the continuous function $f(x) = 90 - x/2$ is -½ in the interval [0, 180]. But the Lipschitz constant L is defined as the supremum of the derivative over the domain; therefore $L = ½$. Hence, the iteration passes the Lipschitz test that $L < 1$, and will converge to the fixed point. From the symmetry of the triangle, the same relationship will hold for the other angles of the triangle; hence $\Delta A_nB_nC_n$ converges towards an equilateral triangle.

For Conjecture 2, let $\angle A = x$ and $\angle B = y$. Then $\angle C = 180^\circ - (x + y)$, and $\angle A_1BC = 90 - y/2$ and $\angle A_1CB = (x + y)/2$, since $\angle A_1BC$ and $\angle A_1CB$ are respectively half the exterior angles at B and C. Thus, $\angle BA_1C = 90 - x/2$, and as in Conjecture 1 we get exactly the same recursive relationship for the successive values of $\angle A_n$, namely: $x_{n+1} = 90^\circ - \frac{x_n}{2}$. And in the same way it follows that $\Delta A_nB_nC_n$ converges towards an equilateral triangle.

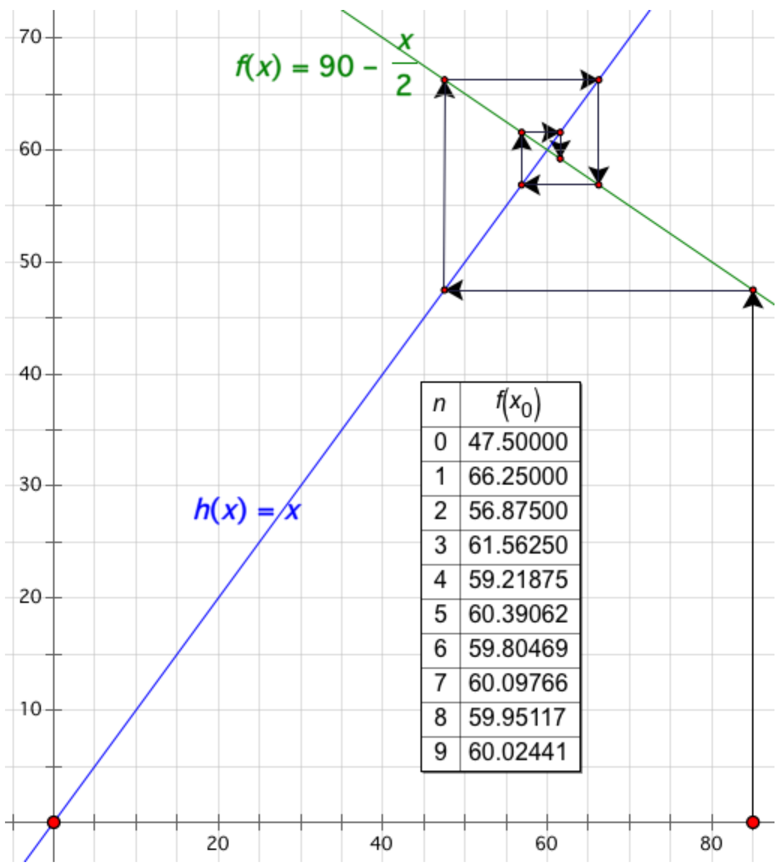
[2] For more information on fixed-point iteration, go to http://en.wikipedia.org/wiki/Fixed_point_iteration



n	$f(x_0)$
0	24.50000
1	77.75000
2	51.12500
3	64.43750
4	57.78125
5	61.10938
6	59.44531
7	60.27734
8	59.86133
9	60.06934

	Step	Start
2	1	15
3	2	82.5
4	3	48.75
5	4	65.625
6	5	57.1875
7	6	61.40625
8	7	59.296875
9	8	60.3515625
10	9	59.8242188

Figure 3: Numerical iteration with *Sketchpad* and *Excel*



n	$f(x_0)$
0	47.50000
1	66.25000
2	56.87500
3	61.56250
4	59.21875
5	60.39062
6	59.80469
7	60.09766
8	59.95117
9	60.02441

Figure 4: Graphical representation of spiraling convergence

Apart from easily carrying out the numerical iteration on a calculator with any starting value, one can also use different software to illustrate the convergence. Figure 3 shows an iteration with *Sketchpad* with a starting value of 131, as well as one with an *Excel* spreadsheet starting with 15.

To create the 1st column in the *Excel* spreadsheet, one can start by putting 1 in A2 and then in A3 type the equation = A2 + 1, and dragging the right bottom corner of this cell down, then fills the column with consecutive numbers. In the 2nd column, one starts by placing 15 in B2, and then in B3 type the equation = 90 - B2/2, and dragging the right bottom corner of this cell down, then fills the column with the calculated values of the iteration.

In practice, it is also useful to sometimes draw graphs of the functions involved as one can then easily visually 'see' whether a particular iteration converges. For example, drawing $y = x$ and $y = 90 - x/2$ as shown in Figure 4 for a starting value of 85, clearly shows how the iteration spirals in towards the fixed point.

It seems reasonable, as the author did, to conjecture that if one iterates the triangle formed by the tangent points of the excircles to the original triangle that it would similarly converge to an equilateral triangle. However, this is not the case as can be seen in Figure 5, and in fact, the iterated triangle eventually degenerates into a straight line.

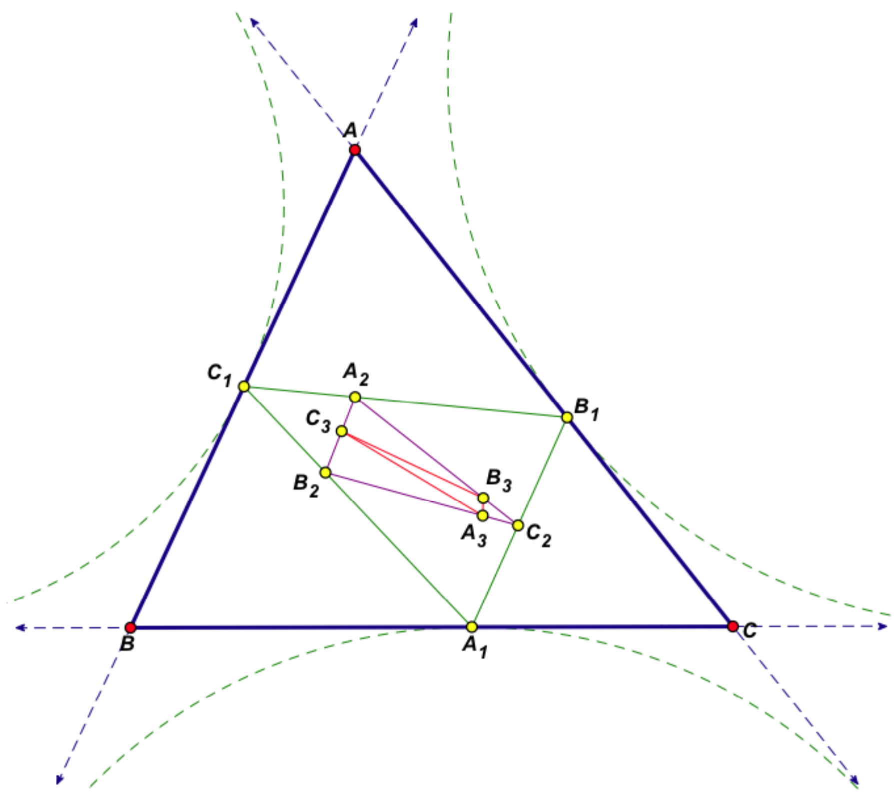


Figure 5: Iteration of tangent points of excircles

Finally, in conclusion, one could argue that in the modern day and age, someone is not mathematically literate if they are completely ignorant of iteration as well as unable to effectively use numerical methods and available computing technology to solve a variety of mathematical problems that cannot be solved otherwise. The book by Choate et al (1999) is particularly suited for an interesting introduction at high school level.

Postscript Notes

1) Shortly after completion of this paper, the author found the result of Investigation 2 mentioned in Johnson (1929, p. 185) though without a formal proof of the convergence.
2) A while later, the author also found the first result mentioned at Wolfram MathWorld at: http://mathworld.wolfram.com/ContactTriangle.html and attributed to Goldoni (2003).

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Ausubel, D.P., Novak, J.D. & Hanesian, H. (1978). *Educational Psychology: A cognitive view*. Holt, Rinehart and Winston: USA.

Choate, J., Devaney, R.L. & Foster, A. (1999). *Iteration: a tool kit of dynamic activities*. Key Curriculum Press: USA[3]. (A limited number of copies of this book as well as the books *Fractals* and *Chaos* are available for customers in Southern Africa from dynamiclearn@mweb.co.za)

De Villiers, M. (1987). Algemene beheersingsvlakke van sekere wiskundige begrippe en werkwyses deur voornemende Wiskunde-onderwysers, *S.A. Tydskrif vir Opvoedkunde*. February, Vol. 7, No. 1, pp. 34-41. (Generals levels of competency and understanding of prospective mathematics teachers of some mathematical concepts and processes).

Goldoni, G. (2003). Problem 10993. *Amer. Math. Monthly*, 110, p. 155.

Johnson, R.A. (1929). *Advanced Euclidean Geometry*. New York: Dover Publications.

[3] A limited number of copies of this book as well as the associated books *Fractals* and *Chaos* are available for customers in Southern Africa from: dynamiclearn@mweb.co.za