Global Journal of Advanced Research
on Classical and Modern Geometries
ISSN: 2284-5569, Vol.12, (2023), Issue 1, pp.158-168

GEOMETRIC PROOFS AND FURTHER GENERALIZATIONS OF DAO THAN OAI'S NAPOLEON HEXAGON THEOREM

HANS HUMENBERGER, BERTHOLD SCHUPPAR, MICHAEL DE VILLIERS

ABSTRACT. Recently we published a paper concerning a generalization of Napoleon's theorem, De Villiers/Humenberger/Schuppar (2022). Then we were pointed to Dao Tanh Oai (2015) where a theorem concerning another generalization of Napoleon's theorem is presented (without a proof). In this paper we present two purely geometric proofs for Dao Tanh Oai's generalization of Napoleon's theorem and some further generalizations.

In Dao Tanh Oai (2015) the following theorem is presented and called Theorem 1, and here we call it the same.

Theorem 1. Let $ABCDEF$ be a hexagon and ΔABG, ΔDHC, ΔIEF all outwardly or inwardly constructed equilateral triangles (in Fig. 1 all outwardly). Let A_1, B_1, C_1 be the centroids of ΔFGC, ΔBHE, and ΔDIA respectively, let A_2, B_2, C_2 be the centroids of ΔDGE, ΔAHF, and ΔBIC respectively. Then $\Delta A_1B_1C_1$ and $\Delta A_2B_2C_2$ are equilateral triangles.
Additionally (not mentioned in Dao Tanh Oai (2015)): There is a third such equilateral $\Delta A_3B_3C_3$ where A_3, B_3, C_3 are the centroids of the triangles IGH, ACE, DBF.

Key words and phrases. Napoleon's theorem, hexagon, equilateral triangles, centroids.

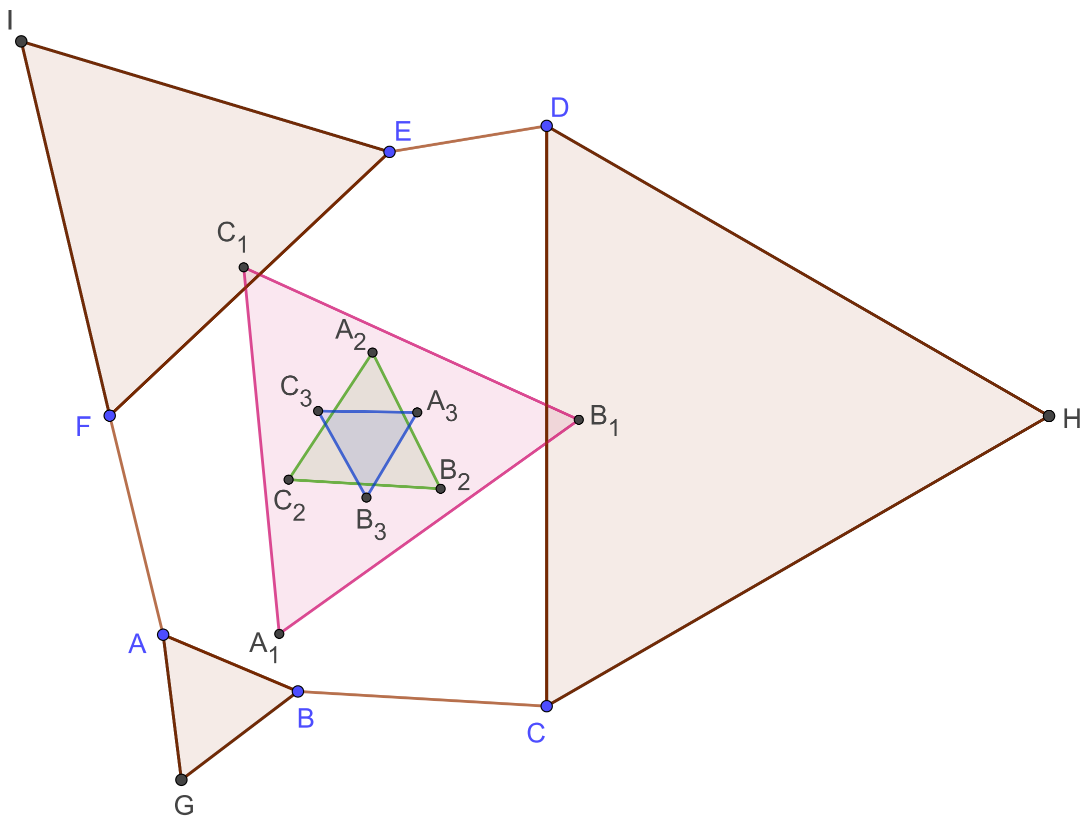


Figure 1

In Dao Tanh Oai (2015) a reference[1] is given where it is said to be a proof for Theorem 1 using *complex numbers*, but we could not find such a proof there. But at this link one can find another very short and dense proof (for an even stronger result) by Bogomolny (2013) using *affine transformations* and *linear algebra*. There no hexagon appears, but rather three arbitrary *directly similar triangles*, and the *equilateral triangles* of Theorem 1 are, of course, directly similar to each other, so that one can immediately see that Theorem 1 is covered by this "final chapter of the asymmetric propeller story", as it is called there.
In the following we give two proofs for Theorem 1[2] which use only geometric arguments. The first proof works with several steps starting from the well-known Napoleon configuration, basically all steps the same, thus a proof for one such step suffices.
The second proof works with the stronger result of the "final chapter of the asymmetric propeller story" but does not use the above mentioned (short, dense, and rather abstract) arguments. We will use another means, which could be called the "fundamental theorem of similarity" (see below). This second proof will also provide a further generalization.

[1]http://www.cut-the-knot.org/m/Geometry/FinalAsymmetricPropeller.shtml
[2]A dynamic web sketch illustrating the theorem, and further generalizations discussed later in this paper, is available for the reader at: dynamicmathematicslearning.com/dao-than-oai-napoleon-generalization.html

Initially, let us briefly recall the well-known configuration of Napoleon's theorem. Let ΔACE be a triangle and on its three sides equilateral triangles (all outwardly or all inwardly, in Fig. 2a all outwardly) are erected. Then the centroids A_1, B_1, C_1 of these equilateral triangles themselves build an equilateral triangle[3]. It is also well known that the line segments AH, CI, and EG are equal and concurrent at the Fermat-Torricelli point T of the initial ΔACE, and that these lines make angles of $60°$ and $120°$ at T (Fig. 2a).

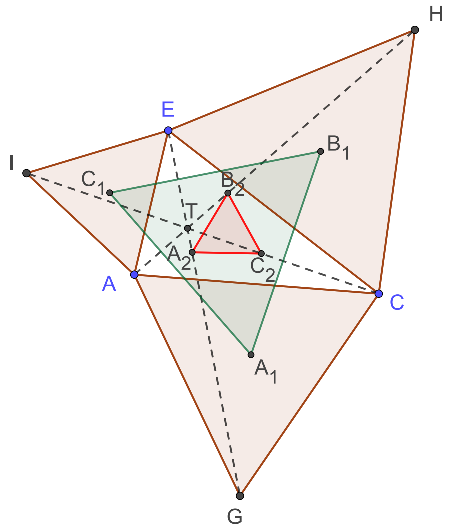
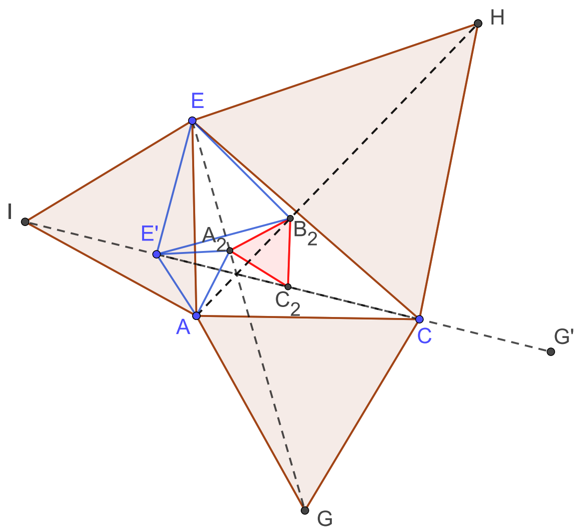


Figure 2a



2b

1. First proof

For this first proof we will need four other results, some of them not so well known, thus we will formulate them as Lemmas and give proofs.

Lemma 1. In Fig. 2a the points A_2, B_2, C_2 which are defined as the points dividing the line segments EG, AH, and CI in the ratio $1:2$ build an equilateral triangle.

Proof. If we rotate the line segment EG around A counterclockwise with $60°$ (yielding the line segment IC) and then again with $60°$ counterclockwise around E we have finally $EG \mapsto AH$ and $A_2 \mapsto B_2$ (see Fig. 2b). Let $E' \in IC$ be the image of A_2 under the first rotation with center A, then $\Delta AA_2E'$ is equilateral and the distance IE' equals one third of IC. Since the image of E' under the second rotation (center E) is B_2 also $\Delta EE'B_2$ is equilateral. But looking at $\Delta EE'B_2$ differently one can say: if we rotate EG around B_2 with $60°$ counterclockwise we have $EG \mapsto E'G'$ (with $G' \in IC$) and the points C_2 and C trisect $E'G'$. Therefore, the image of A_2 under the $60°$ counterclockwise rotation around B_2 is C_2, and thus $\Delta A_2B_2C_2$ is equilateral.

Lemma 2. Let ΔABC be a triangle with centroid G. If C is moved to C' let G' be the centroid of $\Delta ABC'$. Then GG' is parallel to CC' and has one third of its length (Fig. 3a).

[3]One can see immediately that Theorem 1 is a generalization of Napoleon's theorem, because in case of $B = C$, $D = E$, and $F = A$ it is nothing else than Napoleon's theorem.

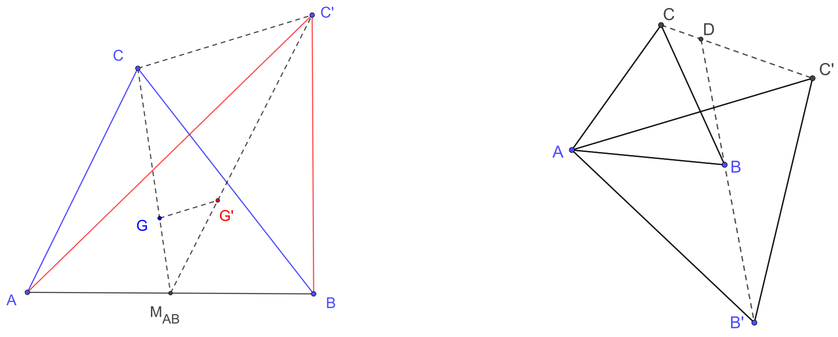


Figure 3a **3b**

Proof: GG' is the image of CC' under the homothety with the midpoint of AB as center and factor $k = \frac{1}{3}$.

Lemma 3. Let ΔABC and $\Delta AB'C'$ be two equilateral triangles sharing the common point A. Then BB' and CC' are equal and make an angle of $60°$ (in Fig. 3b at D [4]).

Proof: The triangles $\Delta ABB'$ and $\Delta ACC'$ are congruent (side-angle-side) and $\angle BAC = 60°$. In other words: $\Delta ACC'$ is $\Delta ABB'$ rotated by $60°$ around A.

Lemma 4. Let ΔABG, ΔDCH, ΔIEF be three equilateral triangles with initially $C = B$ (Fig. 4a; no matter whether or not E and D coincide, analogously with A and F). Moreover, let the centroids A_1, B_1, C_1 of the triangles ΔFCG, ΔBEH, and ΔADI build an equilateral triangle. If then the point C is moved to C', Fig. 4b) also the points A_1, H, and B_1 move ($\rightarrow A_1', H', B_1'$). But still the $\Delta A_1'B_1'C_1$ is equilateral.

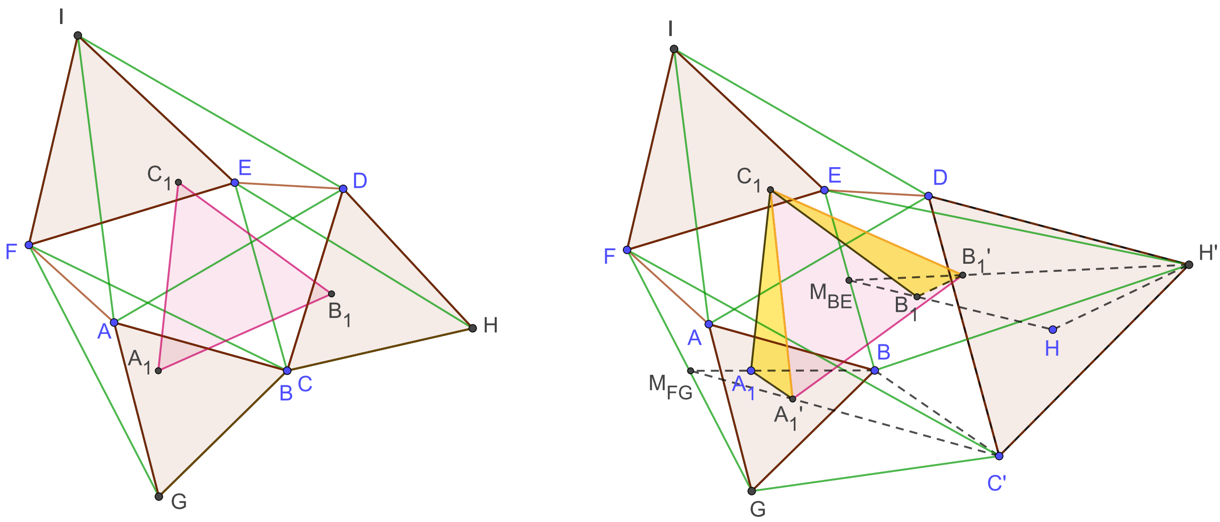


Figure 4a **4b**

[4]This is essentially the same as in the original Napoleon configuration (see Fig. 2a), but here formulated and briefly proven again.

Proof. Due to Lemma 2 we have $A_1A_1'||BC'$ and $A_1A_1' = \frac{1}{3}BC'$ (Fig. 4b). Because of Lemma 3 we have $HH' = BC'$ with an angle of 60° between them. Using again Lemma 2 we know $B_1B_1'||HH'$ and $B_1B_1' = \frac{1}{3}HH'$. Altogether, $A_1A_1' = B_1B_1'$ with an angle of 60° between them, which means that B_1B_1' is A_1A_1' rotated with 60° around C_1. Furthermore, we know C_1B_1 is C_1A_1 rotated with 60° around C_1, thus $\angle C_1A_1A_1' = \angle C_1B_1B_1'$ and the triangles $\Delta C_1A_1A_1'$ and $\Delta C_1B_1B_1'$ are congruent (side-angle-side), and from that we can conclude $C_1A_1' = C_1B_1'$ and $\angle A_1'C_1B_1' = 60°$, or equivalently $\Delta A_1'B_1'C_1$ is equilateral. This completes the proof of Lemma 4.

Now we are prepared to prove Theorem 1. The only thing we have to do is to apply Lemma 4 several times. On the one hand starting with the usual Napoleon configuration ($B = C$, $D = E$, $F = A$) Lemma 4 guarantees that in case $B \neq C$, $D = E$, $F = A$ the $\Delta A_1B_1C_1$ is still equilateral (step 1); in the next step 2 we separate D and E, and in the third step we separate F and A. We wanted to stress the issue of "proof as explanation" (see De Villiers 2012), so we decided to give an illustrative own figure for every step (see Fig. 5a-d). Again, in both steps 2 and 3 due to Lemma 4 the $\Delta A_1B_1C_1$ stays equilateral.

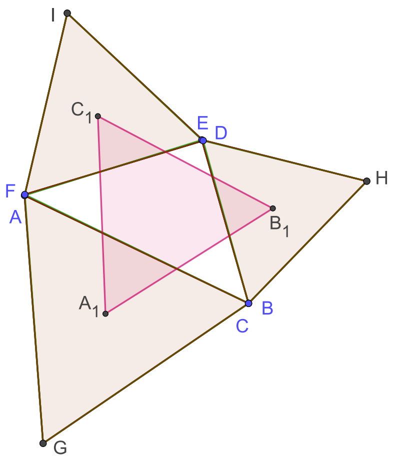
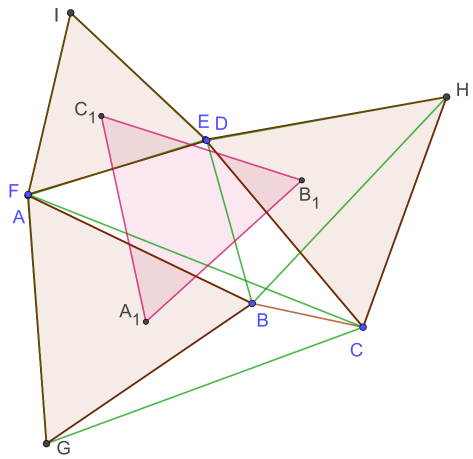


Figure 5a



5b

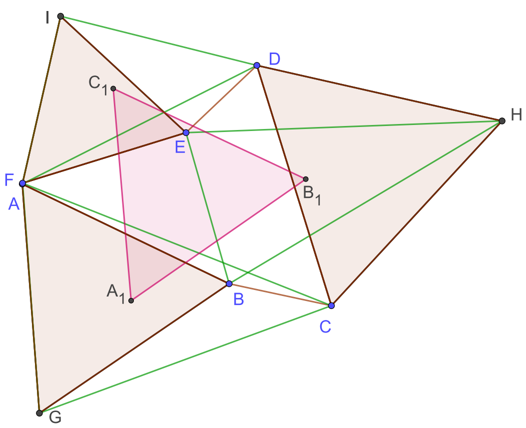
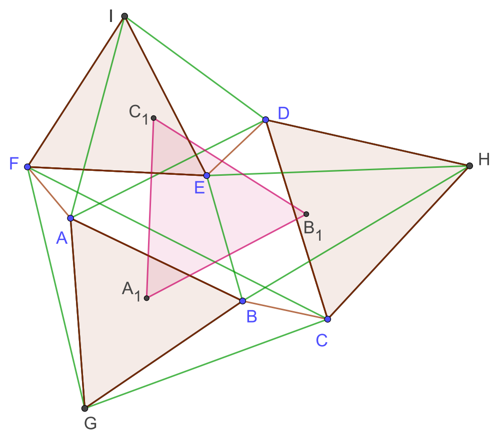


Figure 5c



5d

And with the $\Delta A_2B_2C_2$ we do it the same way starting from the configuration of Lemma 1 which says that the corresponding dividing points in the ratio $1:2$ (namely A_2, B_2, C_2) build an equilateral triangle. Note here that all three "green triangles" (ΔDGE, ΔAHF, ΔBIC) are initially degenerated to line segments, the centroid of ΔEGD is at the beginning the point dividing EG in the ratio $1:2$ (analogous with the others). Then making again three steps (in each step using Lemma 4) yields Fig. 6a-d, the $\Delta A_2B_2C_2$ stays equilateral in each step. This completes the proof of Theorem 1.

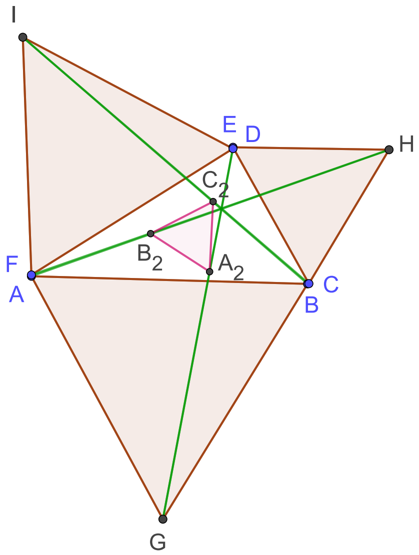
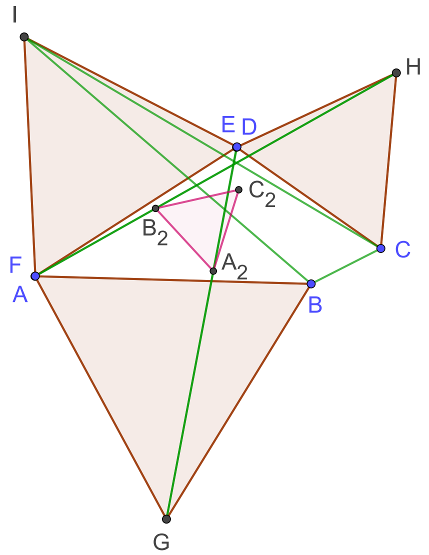


Figure 6a



6b

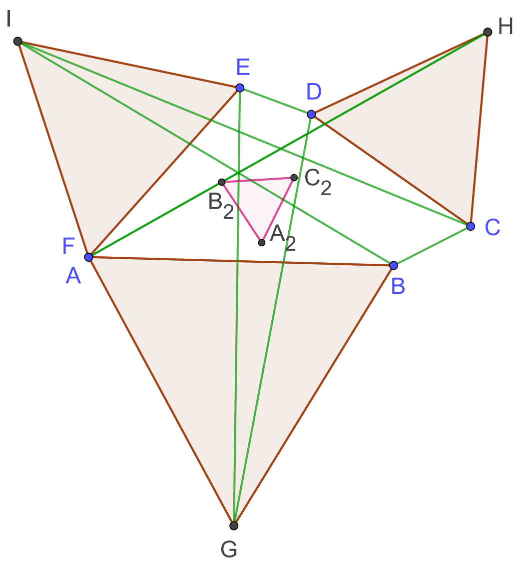
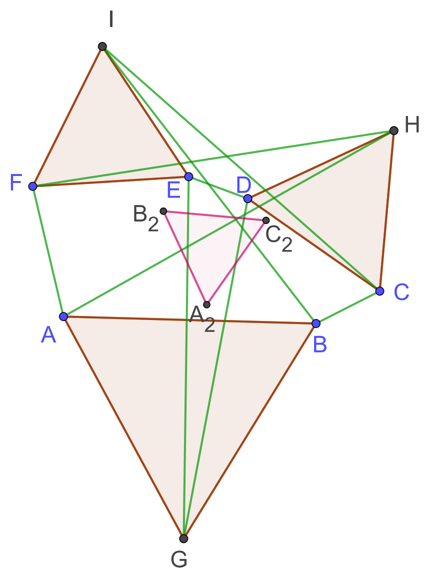


Figure 6c



6d

The proof for the $\Delta A_3B_3C_3$ works analogously. Since the triangles ACE and DBF are the same initially ($A = F$, $B = C$, $D = E$; see Fig. 2a), and since it is well known that in the original Napoleon configuration (Fig. 2a) the centroids of ΔIGH and ΔACE coincide, we know that initially we have $A_3 = B_3 = C_3$, in other words $\Delta A_3B_3C_3$ is an equilateral triangle degenerated to a single point. Again, stepwise separating the points B, C, then E, D, and finally A, F – using Lemma 4 in each step – provides a proof.

2. Second proof

For a second proof we prove a generalization of Theorem 1 to similar triangles. For that we need the following

Fundamental Theorem of Similarity (FTS) [5].
If F and F' are any two directly similar[6] figures with the vertices P in F corresponding to vertices P' in F', and the lines PP' are divided by P'' in the *same* ratio[7], then the new figure F'' formed by the points P'' is directly similar to F and F' (in Fig. 7 F is a quadrilateral $IJKL$).

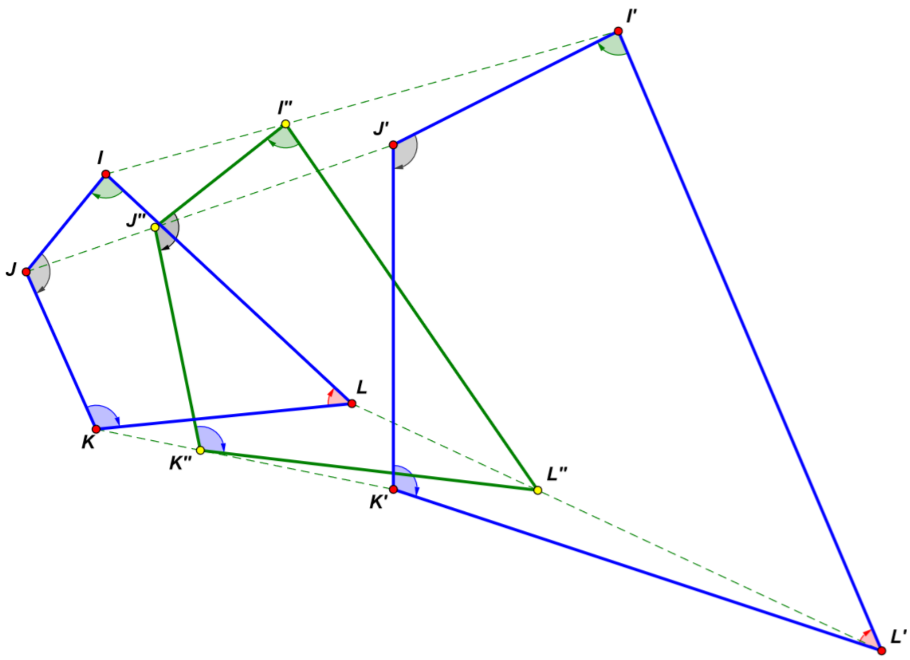


Figure 7

This important and powerful theorem can be proved in various ways (see DeTemple/ Harold (1996), De Villiers (1998), Abel (2007), Fried (2021)). With this result one can prove the "final chapter of the asymmetric propeller story" in a more explanatory way, not using the concepts of *affine operators* and the like.

[5]A dynamic sketch for the case when F is a quadrilateral is available for the reader at http://dynamicmathematicslearning.com/fundamental-theorem-similarity.html.
[6]Two similar figures are *directly similar* if their corresponding angles have the same rotational sense (and are not reversed in relation to each other as in a reflection).
[7]The ratio $r : (1 - r)$ would – in terms of linear algebra – yield $P'' = (1 - r)P + rP'$.

Theorem 2. ("The final chapter of the asymmetric propeller story")
Given three directly similar triangles ABG, DHC, IEF [8] then the centroids A_1, B_1, C_1 of the three triangles (FGC, BHE, DIA) formed by joining corresponding vertices form a $\Delta C_1B_1A_1$ directly similar to the other three.

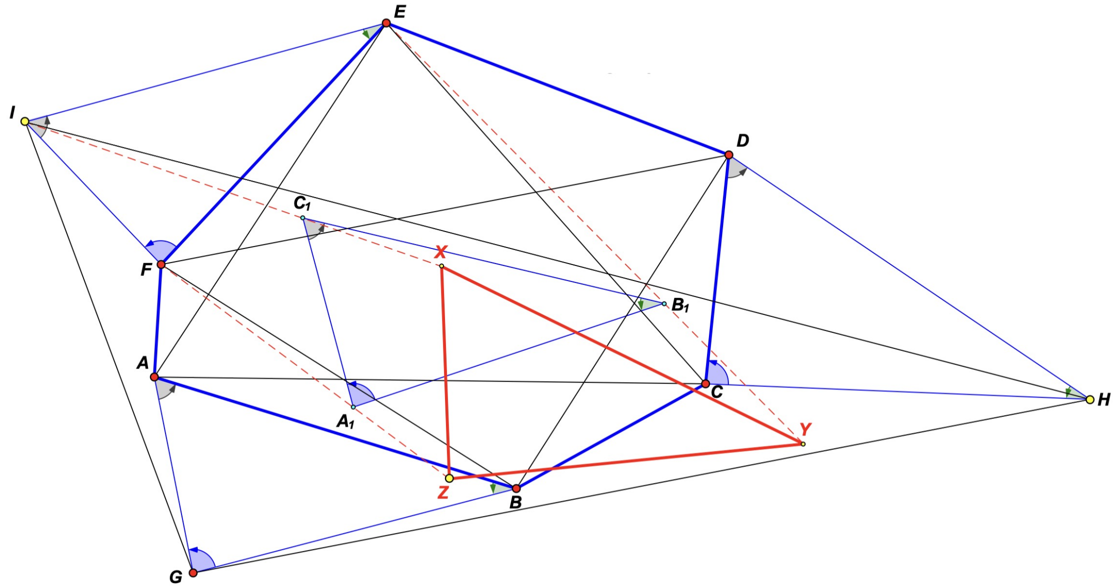


Figure 8

Proof. To come from the similar triangles ABG, DHC, IEF to the resulting similar $\Delta C_1B_1A_1$ we need two steps of FTS: In the first step one combines, say ΔABG and ΔDHC, with factor $r = \frac{1}{2}$ (that means we focus on the midpoints of corresponding points) yielding the ΔXYZ which due to FTS is similar to the triangles ABG, DHC, IEF. Assume a point mass of 1 at each of the vertices $A, B, C, D, E, F, G, H, I$. At each of the vertices X, Y, Z we can now imagine a point mass of 2 (since they are respective midpoints between two corresponding vertices). Thus, in order to get the centroid A_1 of the ΔFGC we have to look for a point dividing ZF in the ratio $1 : 2$ [9], analogously with B_1 and C_1. That means altogether, in the second step one combines ΔXYZ with ΔIEF with factor $r = \frac{1}{3}$ or $r = \frac{2}{3}$ (that depends on which of the two triangles XYZ or IEF one starts from) to get $C_1B_1A_1$ which, again due to FTS, is similar to the triangles ABG, DHC, IEF.

Remarks

- Referring to Theorem 1 all the erected equilateral triangles are, of course, directly similar to each other. Thus, Theorem 2 automatically provides a proof of Theorem 1, for all three cases $\Delta A_iB_iC_i$. All three triangles can be drawn in the same sketch because in each case A_i, B_i, C_i are centroids of triangles built up of *corresponding* vertices of directly similar triangles (Fig. 1).

[8] In Fig. 8 these triangles are erected on the sides of a hexagon $ABCDEF$, in order that this fits to our original hexagon problem. But one could also delete the line segments BC, DE, FA so that the hexagon disappears and only the three directly similar triangles are left.

[9] At F we have a point mass of 1, at Z a point mass of 2; then we use the well known *law of leverage*.

- It should be emphasized that in Theorem 2 and Fig. 8 the order of the vertices in the mentioned triangles is important (the given directly similar triangles and $\Delta C_1 B_1 A_1$): equal angles at the first, second and third vertex, respectively. Analogously, this is important below (alternative arrangements, Fig. 9 and 10).

When the initial three triangles are directly similar (and not equilateral any more), the triangles $A_2B_2C_2$ and $A_3B_3C_3$ (as defined at the beginning) are no longer similar to the initial three similar triangles, because the vertices of the triangle with centroid A_2 are *no longer corresponding* vertices of directly similar triangles (analogously with B_2 and C_2). This brings us to the following paragraph.

Alternative arrangements of directly similar triangles. If three directly similar triangles ABG, HCD & FIE are constructed the centroids A_2, B_2, C_2 of the three triangles (DGE, AHF, BIC) will form a $\Delta B_2C_2A_2$ directly similar to the other three (here the order of vertices is important!) as shown in Fig. 9. But triangles $A_1B_1C_1$ and $A_3B_3C_3$ (in any order of the vertices) would not be similar to the initial three directly similar triangles.

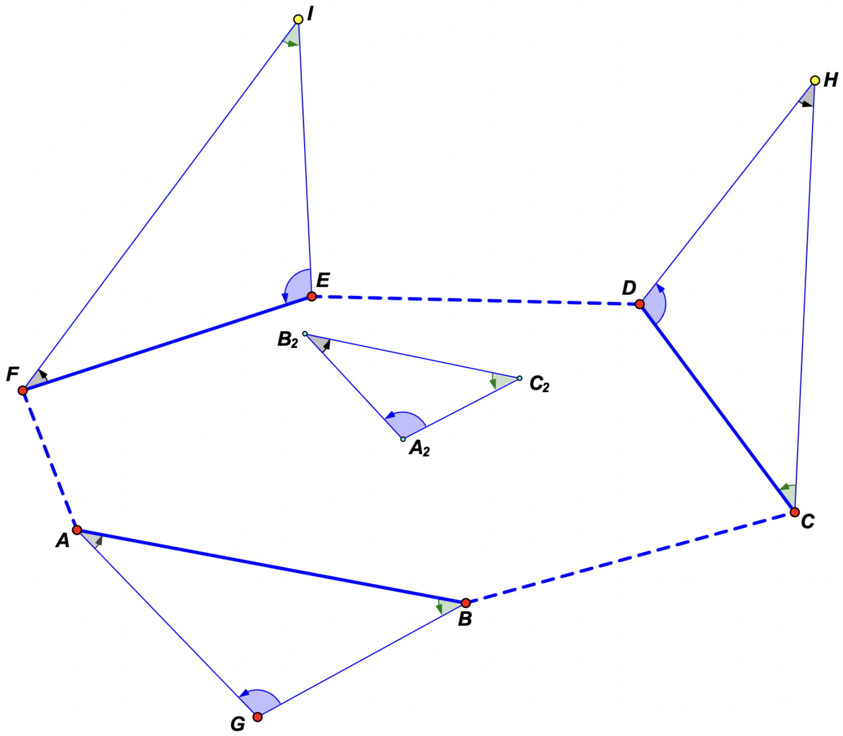


Figure 9

Similarly, if three directly similar triangles ABG, CDH & EFI are given, then the centroids A_3, B_3, C_3 of the three triangles (IGH, ACE & DBF) form a $\Delta B_3C_3A_3$ directly similar to the other three (here the order of vertices is important!) as shown in Fig. 10. Likewise, triangles $A_1B_1C_1$ and $A_2B_2C_2$ (in any order of the vertices) would not be similar to the initial three directly similar triangles.

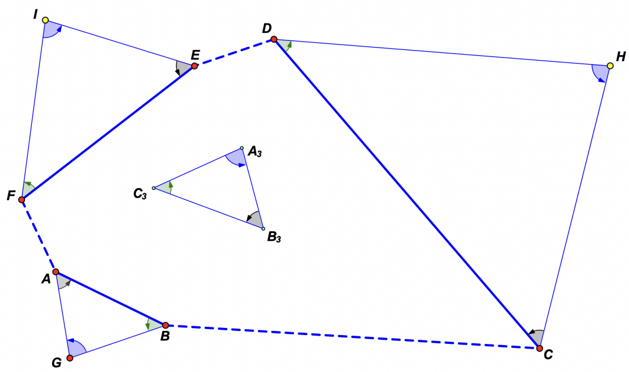


Figure 10

As before, both results in the alternative arrangements above follow from the fundamental theorem of similarity.

Further generalization. With the same technique one can show an even more general result: Given four (n) directly similar quadrilaterals (n-gons) whose corresponding vertices are similarly connected to produce four (n) quadrilaterals (n-gons), and then again taking the centroids (point mass or average) of these four (n) similar quadrilaterals (n-gons) will produce another quadrilateral (n-gon) similar to the other four (n). In Fig. 11 the case of quadrilaterals is shown (A, C, E, G are corresponding vertices and the centroid of the quadrilateral $ACEG$ is K; analogously with L, M, N).
An equivalent formulation would be: Given an octagon ($2n$-gon) and four (n) directly similar quadrilaterals (n-gons) erected on alternate sides of the octagon ($2n$-gon). The corresponding vertices are similarly connected to produce another four (n) quadrilaterals (n-gons), and then taking their centroids (point mass or average) will produce another quadrilateral (n-gon) similar to the other four (n).

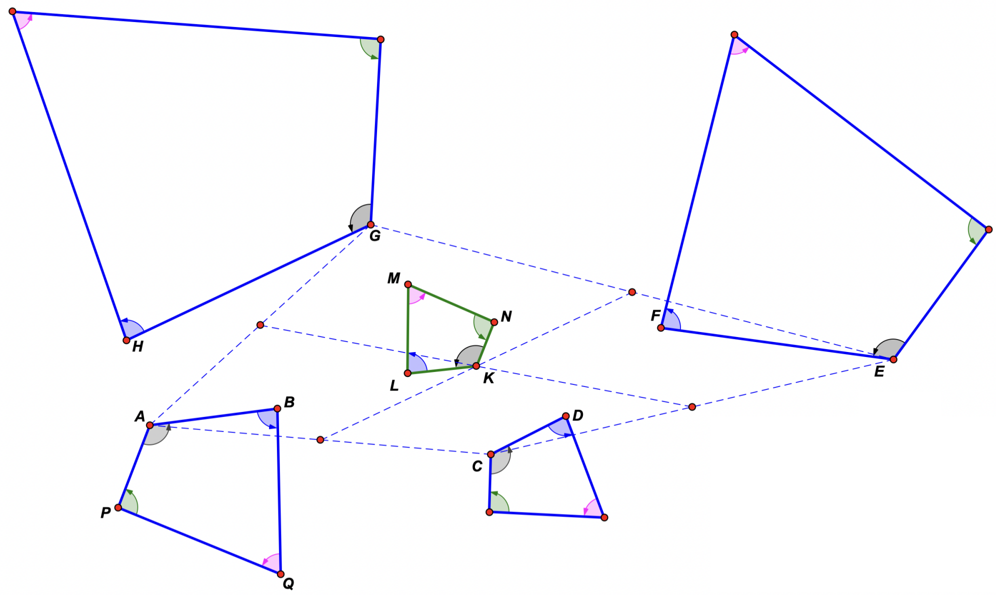


Figure 11

The reader is now invited to further explore other possible arrangements of the directly similar quadrilaterals (n-gons) by considering different cyclic permutations of their labels to find more ways of producing other similar centroid-quadrilaterals (centroid-n-gons).

Conclusion. We presented two geometric proofs of Dao Tanh Oai's generalization of Napoleon's theorem. The first one is a new proof. Moreover, we discussed a further equilateral triangle in this configuration which is not mentioned by Dao Tanh Oai. Though our second proof is not entirely new, and can be seen to be equivalent to that of Bogolmony (2013), we here instead used the Fundamental Theorem of Similarity to produce a more explanatory proof of Dao Tanh Oai's generalization of Napoleon's theorem. In addition, this second proof immediately suggested alternative arrangements of similar triangles and the further generalizations (similar quadrilaterals, or in general, similar n-gons instead of similar triangles). So the second proof also provides a nice illustrative example of the so-called "discovery function" of proof, whereby proving a result gives insight that leads to deeper understanding, other variations, and further generalization (see De Villiers (2012)). Generally, we think geometric proofs are more explanatory than proofs using vectors or complex numbers, and *proof as explanation* is a very important function of proofs (among others like *verification*, see De Villiers (2012)). Proving results in different ways always increases insight both for professional mathematicians as well as for students.

References

[1] Abel, Z. R. *Mean geometry* (2007). http://zacharyabel.com/papers/Mean-Geo_A07.pdf

[2] Bogomolny, A. *A Final Chapter of the Asymmetric Propeller Story* (2013). http://www.cut-the-knot.org/m/Geometry/FinalAsymmetricPropeller.shtml

[3] Dao Tanh Oai. *Two generalizations of the Napoleon theorem* (2015). https://diendantoanhoc.org/index.php?app=core&module=attach&section=attach&attach_id=22546

[4] DeTemple, D., Harold, S. *A round-up of square problems*. Mathematics Magazine, N. 9 (1996), 15-27. https://doi.org/10.1080/0025570X.1996.11996375

[5] De Villiers, M. *Dual generalisations of Van Aubel's theorem*. The Mathematical Gazette, N. 82 (1998), 405-412. https://doi.org/10.2307/3619886

[6] De Villiers, M. *Rethinking proof with Sketchpad*. Emeryville: Key Curriculum Press (2012).

[7] De Villiers, M., Humenberger, H., Schuppar, B. *Jha and Savaran's generalisation of Napoleon's theorem*. Global Journal of Advanced Research on Classical and Modern Geometries (GJARCMG), N. 11 (2022), 190-197. https://geometry-math-journal.ro/pdf/Volume11-Issue2/4.pdf

[8] Fried, M. *From any two directly similar figures, produce a new one*. International Journal of Geometry, N. 10 (2021), 90-94. https://ijgeometry.com/wp-content/uploads/2021/07/8.-90-94.pdf

University of Vienna, Faculty of Mathematics, A – 1090 Vienna, Austria
Faculty of Mathematics, TU Dortmund University, Germany
Faculty of Education (Mathematics), University of Stellenbosch, 7600 Stellenbosch, South Africa
Email address: hans.humenberger@univie.ac.at, berthold.schuppar@tu-dortmund.de, profmd1@mweb.co.za