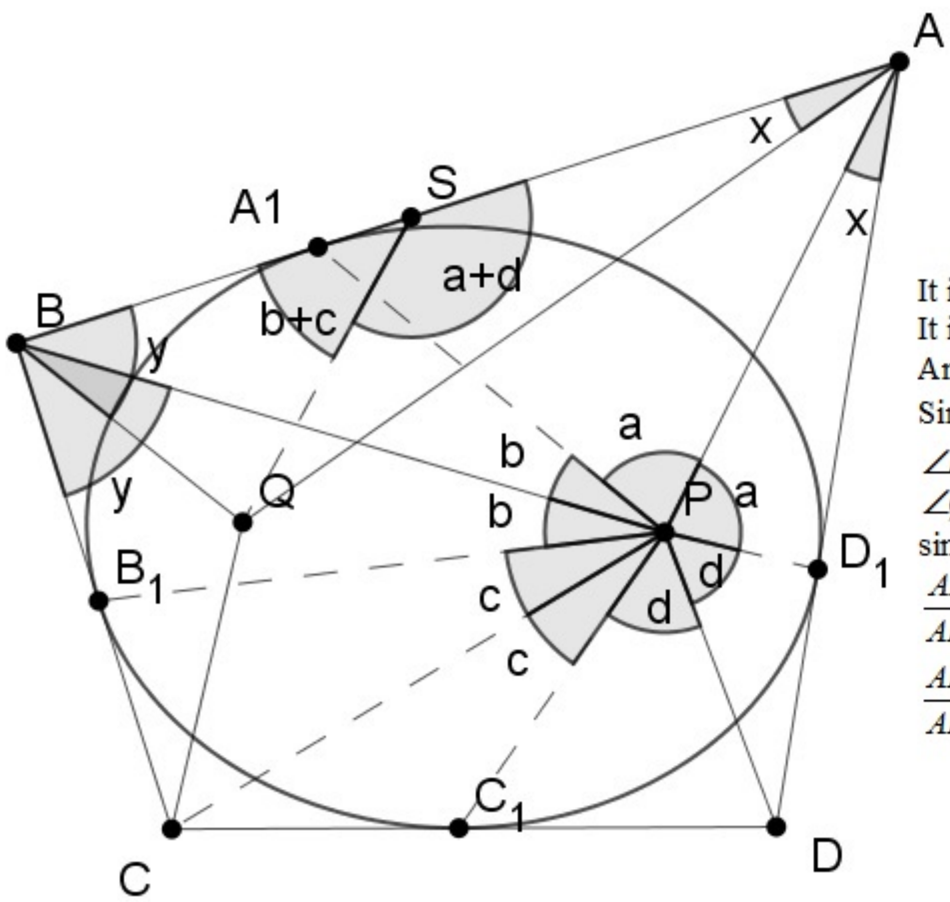
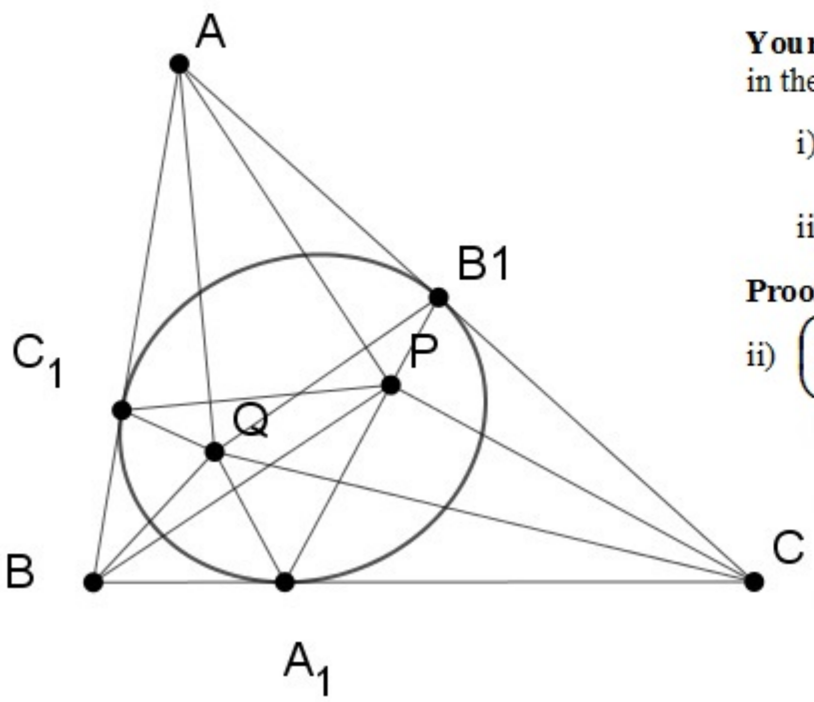


It is known the equality of angles x, and the equality of angles y.
It is known the equality of angles a (Poncelet) (F.G.-M. theorem 889-II)
And similarly the equality of angles b, of angles c, of angles d.
Since $2a+2b+2c+2d=360^\circ \Rightarrow (a+d)+(b+c)=180^\circ$ or $\angle DPA+\angle BPC=180^\circ$. We take on AB the point S such that $\angle QSA=a+d \Rightarrow \angle BSQ=b+c$. The triangles PAD, SAQ are similar and the same for the triangles BSQ, BPC. So from

$$\frac{AP}{AD}=\frac{AS}{AQ} \;\&\; \frac{BP}{BC}=\frac{BS}{BQ} \Rightarrow \frac{AP\cdot AQ}{AD}+\frac{BP\cdot BQ}{BC}=AS+BS=AB \quad \text{or}$$

$$\frac{AP\cdot AQ}{AD\cdot AB}+\frac{BP\cdot BQ}{BC\cdot AB}=1.$$ Similary we get $$\frac{CP\cdot CQ}{CD\cdot CB}+\frac{DP\cdot DQ}{DC\cdot DA}=1.$$



Your generalization theorem 1: If an ellipse with foci P, Q is inscribed in the convex quadrilateral $ABCD$ then

i) for adjacent vertices we have $\dfrac{AP\cdot AQ}{AB\cdot AD}+\dfrac{BP\cdot BQ}{BC\cdot BA}=1$

ii) for opposite vertices we have $\dfrac{AP\cdot AQ}{AB\cdot AD}=\dfrac{CP\cdot CQ}{CB\cdot CA}$

Proof: i) The proof is given

ii) $\left(\dfrac{AP\cdot AQ}{AB\cdot AD}+\dfrac{BP\cdot BQ}{BC\cdot BA}=1,\ \dfrac{BP\cdot BQ}{BC\cdot BA}+\dfrac{CP\cdot CQ}{CB\cdot CA}=1\right)\to\dfrac{AP\cdot AQ}{AB\cdot AD}=\dfrac{CP\cdot CQ}{CB\cdot CA}$.

Corollary theorem 2 (Gregory's theorem and more)

If an ellipse with foci, P, Q is inscribed in ABC at the points A_1, B_1, C_1 then

i) $\dfrac{AP\cdot AQ}{AB\cdot AC}=\dfrac{A_1P\cdot A_1Q}{A_1B\cdot A_1C}$

ii) $\dfrac{AP\cdot AQ}{AB\cdot AC}+\dfrac{BP\cdot BQ}{BC\cdot BA}+\dfrac{CP\cdot CQ}{CA\cdot CB}=1$

iii) $\dfrac{A_1P\cdot A_1Q}{A_1B\cdot A_1C}+\dfrac{B_1P\cdot B_1Q}{B_1C\cdot B_1A}+\dfrac{C_1P\cdot C_1Q}{C_1A\cdot C_1B}=1$

Proof. i) Since ABA_1C is a degenerated quadrilateral from theorem 1ii we get $\dfrac{AP\cdot AQ}{AB\cdot AC}=\dfrac{A_1P\cdot A_1Q}{A_1B\cdot A_1C}$.

ii) $\left(\dfrac{AP\cdot AQ}{AB\cdot AC}+\dfrac{BP\cdot BQ}{BA\cdot BA_1}=1,\dfrac{AP\cdot AQ}{AB\cdot AC}+\dfrac{CP\cdot CQ}{CA\cdot CA_1}=1\right)$ (1) $\to\dfrac{BP\cdot BQ}{BA\cdot BA_1}=\dfrac{CP\cdot CQ}{CA\cdot CA_1}=m$ that means

$\dfrac{BP\cdot BQ}{BC\cdot BA}+\dfrac{CP\cdot CQ}{CA\cdot CB}=\dfrac{mBA_1}{BC}+\dfrac{mCA_1}{BC}=m$ or from (1) we conclude Gregory's theorem

$$\frac{AP\cdot AQ}{AB\cdot AC}+\frac{BP\cdot BQ}{BC\cdot BA}+\frac{CP\cdot CQ}{CA\cdot CB}=1.$$

iii) From i) & ii) since we have other 2 degenerated quadrilaterals BCB_1A, CAC_1B we get

$$\frac{A_1P\cdot A_1Q}{A_1B\cdot A_1C}+\frac{B_1P\cdot B_1Q}{B_1C\cdot B_1A}+\frac{C_1P\cdot C_1Q}{C_1A\cdot C_1B}=1.$$