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An Investigation of Some Properties of the General Haag Polygon

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Introduction

The famous Dutch artist, M C Escher (1898-1972) investigated in his notes (see Schattschneider, 1990, p. 90) a tiling of the plane first mentioned in a paper by Haag in 1923 with a specific type of congruent non-regular hexagon, called a Haag hexagon by John Rigby (2003). The construction of the basic tile with some of its properties is usually stated as follows:

> Let *ABC* be an equilateral triangle and *F* any point. Choose point *D* so that $AF = AD$ and $\angle DAF = 120°$. Choose *E* so that $BE = BD$ and $\angle DBE = 120°$. Then $CE = CF$ and $\angle ECF = 120°$.

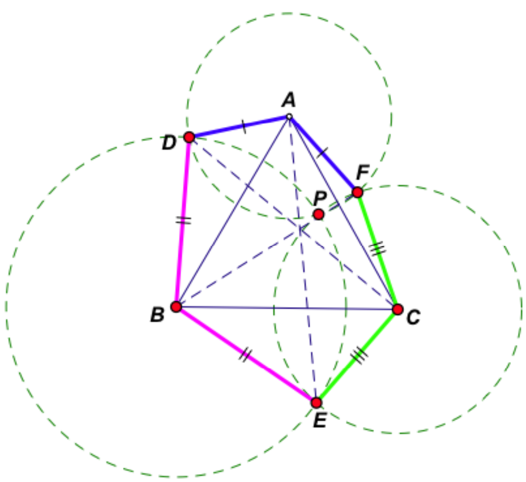

Figure 1: Construction of Haag hexagon

Schattschneider (1990, p. 90) mentions that Escher's son, George, already pointed out that one can construct the Haag hexagon differently by starting with any triangle *DEF* and then constructing on each of its sides similar isosceles triangles with vertex angles of 120° (at *A*, *B* and *C*). However, we can also construct the Haag hexagon in another way as follows (see Figure 1, above):

> Construct equilateral $\triangle ABC$. Draw a circle with centre *A* and arbitrary radius *AD*. Construct a circle with centre *B* and radius *BD*. Label as *P* the other intersection of the two circles centred at *A* and *B*. Draw a circle with centre *C* and radius *CP*. Label as *E* and *F* respectively the other intersections of circle *C* with circles *B* and *C*. Then $\angle DAF = \angle DBE = \angle ECF = 120°$ and *ADBECF* is a Haag tile.

Proof

Quadrilaterals *ADBP* and *AFCP* are both kites. Hence, $\angle DAB = \angle PAB$ and $\angle FAC = \angle PAC$. But $\angle PAB + \angle PAC = 60°$; hence $\angle DAF = \angle DAB + \angle PAB + \angle FAC + \angle PAC = 120°$. Similarly, for the other two angles $\angle DBE$ and $\angle ECF$.

Tiling with the Haag hexagon

To create a tiling with the Haag hexagon, we can start with a basic tiling of the plane with equilateral triangles, and then cover the plane by rotating the Haag hexagon by 120° and/or 240° around each vertex of the equilateral triangular grid (see Figure 2).

It is not clear whether Escher actually used the Haag tiling to create some of his art, but it is suggestive that some of his work such as the 'running men' displays the same rotational symmetry as the Haag tiling. [This drawing is officially known as 'Symmetry Drawing E21'; an image can be found at

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www.pinterest.com/staciekearns/mc-escher/]

Though a grid of equilateral triangles is faintly visible, Rigby (2003, p. 422) points out that we cannot be sure whether Escher actually used it in his construction of the tiling or whether it was perhaps introduced later for the production of his eventual lithograph.

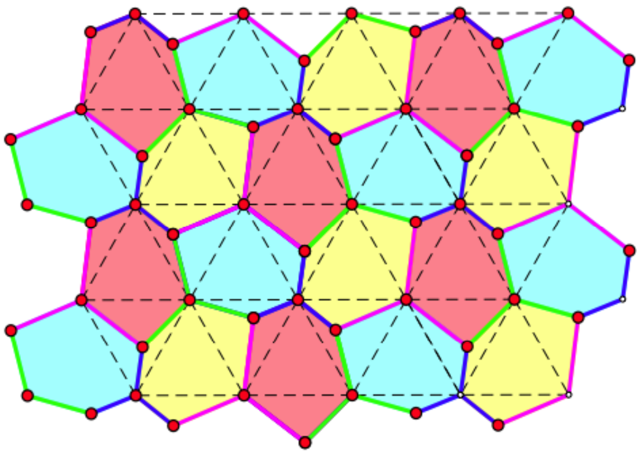

Figure 2: Haag tiling

A concurrency property of the Haag hexagon

Escher mentions without proof in his notes that the diagonals *AE*, *BF* and *DC* of the Haag hexagon *ADBECF* are concurrent (as illustrated in Figure 1). Rigby (1991 & 2003) elegantly shows not only how one could use Napoleon's theorem to prove this result, but also vice versa, how one could use the tiling to prove Napoleon's theorem as well as Fermat's triangle theorem.

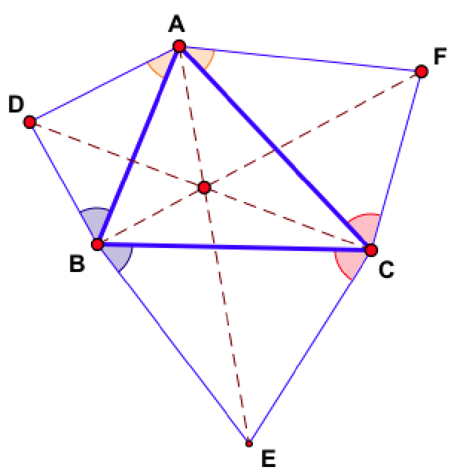

Figure 3: General concurrency theorem

From the earlier mentioned viewpoint of constructing similar isosceles triangles with vertex angles of 120° on the sides of ΔDEF to create the Haag hexagon, the result can be seen to follow immediately from a more general theorem proved in De Villiers (1995), and with different proofs in earlier papers by other authors, namely: "If triangles *DBA*, *ECB* and *FAC* are constructed outwardly or inwardly on the sides of any ΔABC so that $\angle DAB = \angle CAF$, $\angle DBA = \angle CBE$ and $\angle ECB = \angle ACF$ then *AE*, *BF* and *DC* are concurrent (see Figure 3)."

Generalizing the Haag hexagon to a Haag polygon

The Haag hexagon can be generalized to any hexagon *ADBECF* with the properties shown in Figure 4, but apart from the interesting property that its diagonals *AE*, *BF* and *DC* are concurrent, it does not in general tile, as far as I'm aware, or have any other interesting properties.

For this investigation we'll instead be looking at applying the Haag 'circle' construction mentioned above to a general triangle as well as to other polygons such as quadrilaterals, hexagons, etc. to create what I'm choosing to define as a 'Haag polygon' and to explore some of its general and specific properties. The mathematical results discussed here are elementary and could be a suitable investigative activity for high school learners and teachers, giving them an opportunity to apply some basic geometric properties and theorems in a novel context. The reader is invited to dynamically

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explore some of the properties discussed below at:
http://dynamicmathematicslearning.com/haag-hexagon-tiling.html

The general triangle case

The Haag 'circle' construction done on a general triangle is shown in Figure 4. Though the general case does not tile and its main diagonals are not concurrent, it still has the Haag property that, irrespective of the position of point D, $\angle DAF = 2\angle BAC$, $\angle DBE = 2\angle ABC$ and $\angle ECF = 2\angle BCA$.

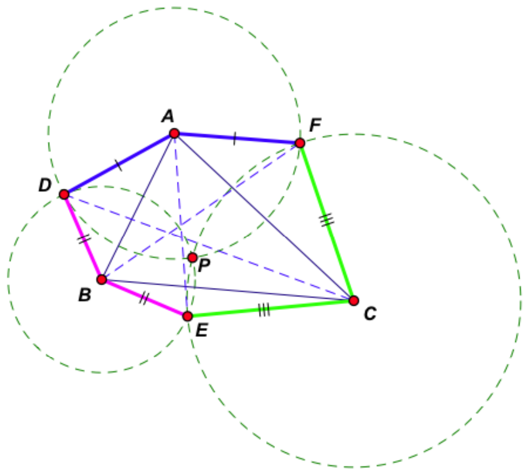

Figure 4: General Haag hexagon

The proof of the angle relationships follows as before directly from the formed kites *ADBP*, *BECP* and *APCF*, and is left to the reader.

However, if for the 'circle' construction for a general triangle we let P coincide with the circumcentre O of ΔABC, then *ADBECF* becomes a hexagon with all sides equal and opposite sides parallel as shown in Figure 5 (and can therefore tile the plane by means of translations).

Proof

Since P coincides with the circumcentre, all three circles are now of equal size and the sides of the hexagon are equal.

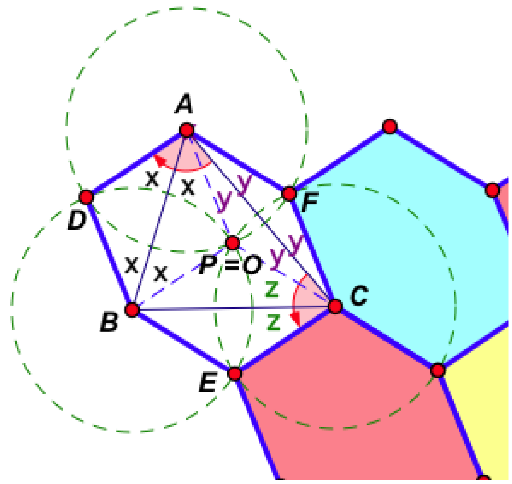

Figure 5: Haag hexagon with equal sides and opposite sides parallel

Moreover, if we mark the equal angles formed by the three rhombi in Figure 5 as indicated, then

$$\angle DAC + \angle ECA = 2x + 2y + 2z.$$

But the angle sum of $\Delta ABC = 2x + 2y + 2z = 180°$. Hence, $\angle DAC + \angle ECA = 180°$, and since they are co-interior angles, it follows that $AD \| CE$. Similarly, the other two pairs of opposite sides can be shown to be parallel.

The quadrilateral case

When applying the Haag 'circle' construction to a quadrilateral *ABCD*, we obtain a Haag octagon *AEBFCGDH* which has the same angle relationships as the general Haag hexagon, e.g. $\angle EAH = 2\angle BAD$, etc., but it unfortunately does not tile or has the diagonal concurrency property.

Of interest though is that for a trapezium *ABCD* with $AD \| BC$ as shown in Figure 6 below, the diagonal *FH* passes through P, the point of concurrency of the circles.

Proof 1

Connect H to P and P to F. It is now required to prove that *HPF* is a straight line. Quadrilateral *AHDP* is a kite. Hence, $PH \perp AD$. In the same way $PF \perp BC$. Draw a line through P parallel to *AD* and *BC*. It then follows that adjacent co-interior angles at P are 90°, from which it follows that $\angle HPF = 180°$.

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Proof 2

Alternatively, we can use the properties of circles. Again connect *H* to *P* and *P* to *F* with the same requirement to prove that *HPF* is a straight line. The marked exterior $\angle EAH = 360° - 2\angle BAD$. From the angle at the centre theorem, we have $\angle EPH = 180° - \angle BAD$. Similarly, it follows that $\angle EPF = 180° - \angle ABC$. Since $\angle BAD$ and $\angle ABC$ are supplementary, it follows that

$$\angle EPH + \angle EPF = 360° - (\angle BAD + \angle ABC) = 180°.$$

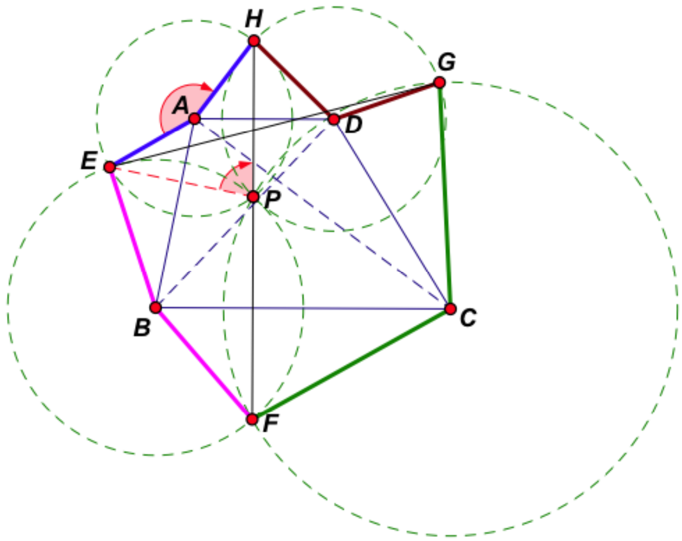

Figure 6

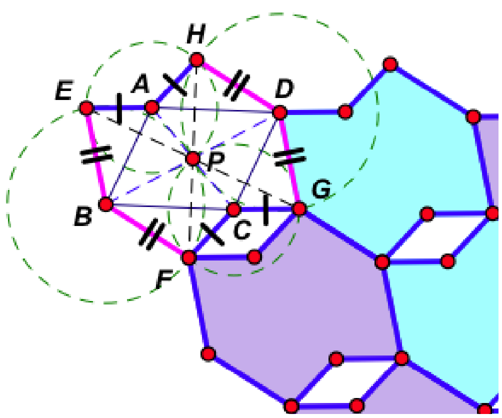

Figure 7: Tiling with a half turn symmetric Haag octagon and rhombus

From the result in Figure 7, it follows that starting with a parallelogram *ABCD* instead of a trapezium, the diagonals *EG* and *FH* of the Haag octagon, both pass through *P*, the point of concurrency of the circles. If we further let *P* coincide with the point of symmetry of the parallelogram the Haag octagon also has half turn symmetry. We can also create a tiling of the plane using translations with this Haag octagon together with a rhombus to fill in the gaps as shown in Figure 7.

If we start with a rectangle or square as shown in Figure 8 on the next page, the sides *HA* and *AE*, etc. of the Haag hexagon *AEBFCGDH* become collinear because $\angle EAH = 2\angle BAD = 180°$, etc. A degenerate quadrilateral *EFGH* is therefore formed, and which obviously can tile the infinite plane. More over, from the result in Figure 6, the diagonals *EG* and *FH* are not only concurrent with the point of concurrency of the four circles, but also perpendicular to each other.

The pentagon and hexagon cases

Starting with the general pentagon and hexagon, do not produce Haag polygons with any additional properties except the angle invariance already mentioned in relation to Figure 4.

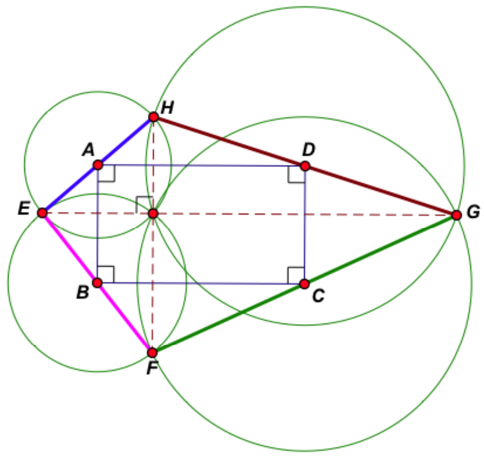

Figure 8: A degenerate Haag octagon forming a quadrilateral

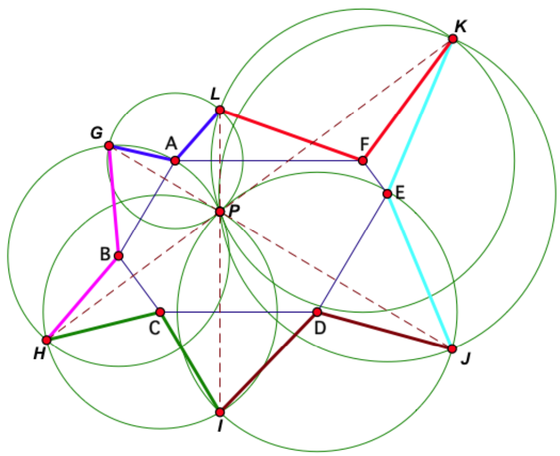

Figure 9: A Haag dodecagon with diagonals *GJ*, *HK* and *IL* concurrent

However, perhaps of special interest is that if we start with a hexagon *ABCDEF* with opposite sides parallel, then similarly to the parallelogram case, the diagonals *GJ*, *HK* and *IL* of the formed Haag dodecagon are concurrent with *P*, the point of concurrency of the generating circles (see Figure 9). A proof similar to the ones in Figure 6 can be given, and is left as an exercise to the reader. It is also easy to see that the same argument would hold when starting with any octagon, decagon, etc. with opposite sides parallel.

References

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