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Mathematical Treasure Hunting

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This Sketchpad presentation involves an interesting problem which can easily be explained (proved) with transformation geometry. The explanation provides insight into why it is true, leading to an immediate generalization, thus illustrating the discovery function of logical reasoning (proof).

Introduction

Consider the following interesting treasure hunt problem from Srinivasan (1983):
"*A man finds an old document left by his grandfather. The document gives specific directions for reaching a certain island. On this island, he has to locate a specific oak tree, a specific pine tree and a gallows. He is asked to start from the gallows and march up to the oak tree counting the number of steps needed to reach the oak tree. At the oak tree, he must turn to the right and march the same number of steps. At this spot he must put a spike in the ground. He must then return to the gallows and then march to the pine tree counting the number of steps needed to reach the pine tree. At the pine tree, he must now turn to the left and march the same number of steps. At the second spot he has to put another spike in the ground. He must then dig at the midpoint of the line segment connecting the two spikes to recover the treasure buried there.*

Upon reaching the island, the man easily finds the specific oak tree and the specific pine tree. However, much to his dismay, he cannot locate the gallows. Rain and decay completely obliterated any traces of the place where the gallows once stood. Can the man still find the treasure? If so, how?"

Most surprisingly, the position of the treasure is completely independent of the position of the gallows, and entirely determined by the positions of the two trees! In other words, the man could start from any arbitrary position for the gallows. How is this possible? It seems quite impossible and counter-intuitive. Perhaps the reader may even doubt this seemingly preposterous claim. If so, the reader is strongly encouraged at this point to test the claim by making an appropriate construction in a dynamic geometry environment such as *Sketchpad* or *Cabri*.

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Formulation

Mathematically, the problem can be formulated as follows: Let A and B be distinct points in the plane (see Figure 1). Start from an arbitrary point P in the plane. After reaching the point A from P, turn to the right at a right angle and mark of the point S_1 such that PA = AS_1. Proceed from P to point B and turn to the left at right angles and mark off the point S_2 such that PB = BS_2. Show that the midpoint T of segment S_1S_2 is independent of the position of P.

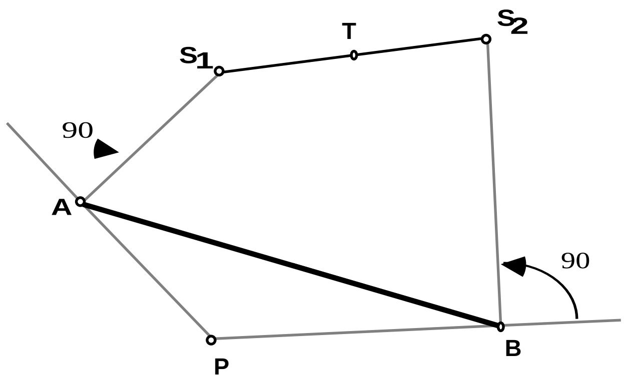

Figure 1

In principle, it is possible to use coordinate geometry to show that the midpoint T (i.e. the treasure) is independent from the position of P (i.e the gallows). However, this is likely to involve a lot of tedious calculation. Alternatively, complex numbers could be used as illustrated by Srinivasan (1983) or one could consider vectors. Personally such methods, although powerful, are somewhat algorithmic, and not very explanatory of why a result is true. The purpose of this article is to provide a purely geometric proof that not only provides greater insight into why the result is true, but also leads to some generalisations.

Proof

The result follows directly from the following very useful, theorem from transformation geometry which deserves to better known:
"The sum of two rotations with centres A and B through angles a and b respectively is a rotation through the angle $a + b$ around some centre O."

Consider Figure 2. The sum of the two rotations carries the centre A of the first into a point A' such that A'B = AB and angle ABA' = b. (The first rotation leaves A in place, and the second carries A into A'). The sum of the two rotations also carries a point B' into B such that B'A = BA and angle BAB' = a. (The first rotation carries B' into B and the second leaves B in place).

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From this it follows that the centre O we are seeking is equidistant from B and B' and from A' and A; consequently it can be found as the point of intersection of the perpendicular bisectors l_1 and l_2 of the segments BB' and A'A respectively. But from the figure, it is clear that l_1 passes through A and angle OAB = $a/2$, and that l_2 passes through B and angle ABO = $b/2$. The lines l_1 and l_2 are completely determined by these conditions; therefore we have found the desired centre of rotation O. Finally, it should now be clear that angle BOB' = $a + b$ = angle AOA' so that a rotation of $a + b$ around O maps B' onto B and A onto A'.

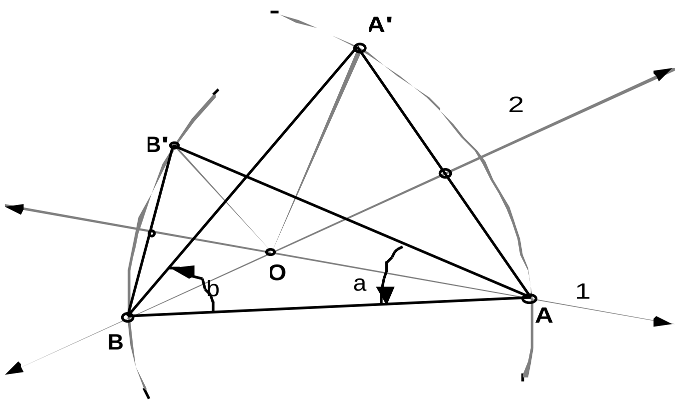

Figure 2

Let us now return to our original problem by considering Figure 3. As shown, the original configuration is equivalent to two clockwise rotations, namely, a clockwise rotation of 90 degrees of S_1 around A followed by another clockwise rotation of 90 degrees of P around B. Therefore S_2 is the image of S_1 by the two rotations around A and B. But as we have seen from the theorem above, the sum of these two rotations is equivalent to a half turn (180 degrees), whose center must be located at the midpoint of S_1S_2. Since this center of the half turn is completely determined as shown above by points A and B, and the sum of the two rotations, it follows that T is fixed, and therefore independent of P.

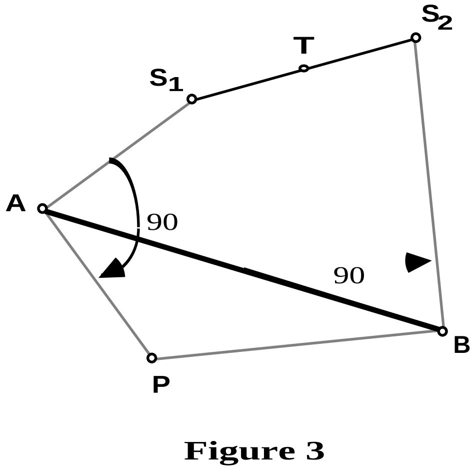

Figure 3

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To locate the treasure buried at T, the man could therefore choose any point P as the gallows to start from, and carry out the prescribed procedure. On the other hand, from the above theorem, the position of T could also be obtained directly from the configuration shown in Figure 4, an adaptation of Figure 2 for this particular case. It follows easily that B'ABA' is a rectangle with T located at the intersection of its diagonals. If X is therefore the midpoint of AB, T could also be located simply by constructing segment XT perpendicular to AB (in the appropriate direction), and equal to AB/2.

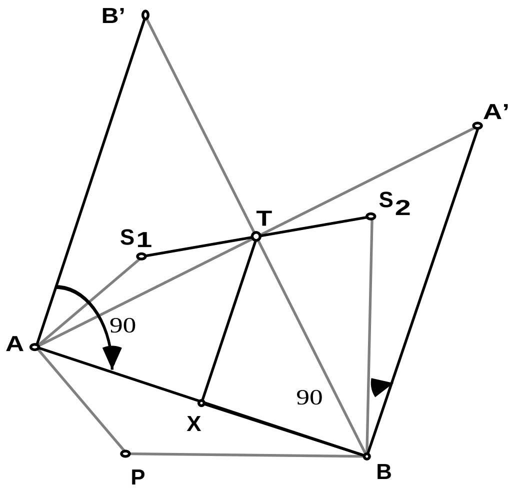

Figure 4

From the above theorem, it also follows immediately that T would be independent of P for any two angles of rotation, provided they sum to 180 degrees. In fact, the result can be generalised to three (or more) rotations at points A, B and C as shown in Figure 5, provided the angles *a*, *b* and *c* sum to 180 degrees. (In terms of the original problem, one could start from any arbitrary point P, turn right through 180 - *a* at A, left at B through 180 - *b*, and left at C though 180 - *c*).

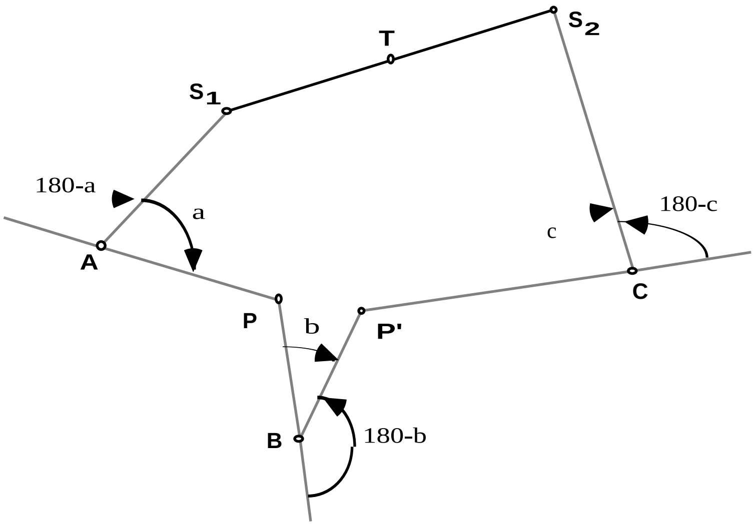

Figure 5

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This is also an excellent example of how the production of a proof can lead to a further generalization by providing insight into why the result is true (compare with the *discovery* function of proof in De Villiers, 1990).

Further Generalisation

The result that T is independent of P can even be further generalised by using the idea of a spiral similarity, denoted by (k, z), and which is the sum (composition) of a dilation with factor k and a rotation of angle z about a fixed center. Now consider the following result from De Villiers (1996: 105-106):

"The sum of two spiral similarities (k, a) and $(1/k, b)$ around centers A and B is a half-turn, if $a + b = 180$ degrees."[1]

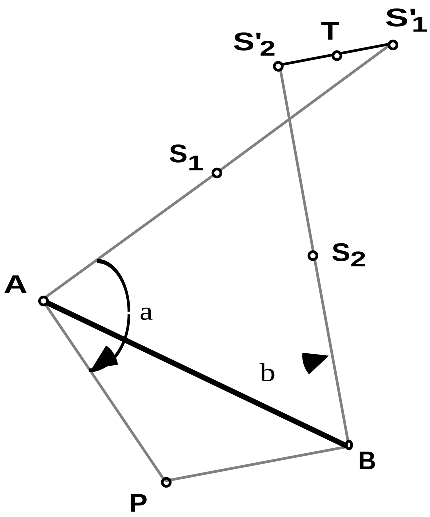

Figure 6

Using spiral similarities, we could therefore further generalize the result so that $AS'_1 = AP/k$ and $BS'_2 = BP/k$. An example is shown in Figure 6 where the dilation factor $k = 1/2$.

Note

[1]This result is a special case of the following more general theorem proved in Yaglom (1968: 40-42):

"The sum of two spiral similarities (k, a) and (m, b) around centers A and B is a spiral similarity $(km, a + b)$."

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