# Geometry

**G1.** The points $P$ and $Q$ are chosen on the side $BC$ of an acute-angled triangle $ABC$ so that $\angle PAB = \angle ACB$ and $\angle QAC = \angle CBA$. The points $M$ and $N$ are taken on the rays $AP$ and $AQ$, respectively, so that $AP = PM$ and $AQ = QN$. Prove that the lines $BM$ and $CN$ intersect on the circumcircle of the triangle $ABC$.

**Solution 1.** Denote by $S$ the intersection point of the lines $BM$ and $CN$. Let moreover $\beta = \angle QAC = \angle CBA$ and $\gamma = \angle PAB = \angle ACB$. From these equalities it follows that the triangles $ABP$ and $CAQ$ are similar (see Figure 1). Therefore we obtain

$$\frac{BP}{PM} = \frac{BP}{PA} = \frac{AQ}{QC} = \frac{NQ}{QC}\,.$$

Moreover,

$$\angle BPM = \beta + \gamma = \angle CQN\,.$$

Hence the triangles $BPM$ and $NQC$ are similar. This gives $\angle BMP = \angle NCQ$, so the triangles $BPM$ and $BSC$ are also similar. Thus we get

$$\angle CSB = \angle BPM = \beta + \gamma = 180^\circ - \angle BAC\,,$$

which completes the solution.

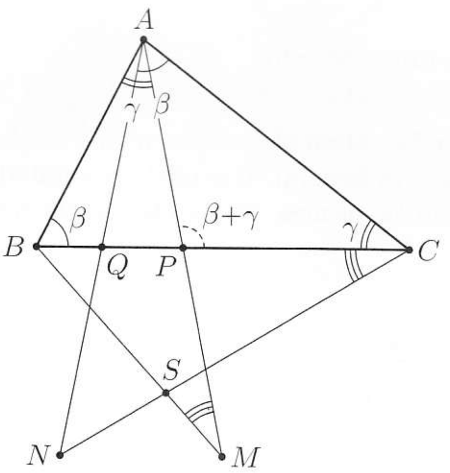

Figure 1

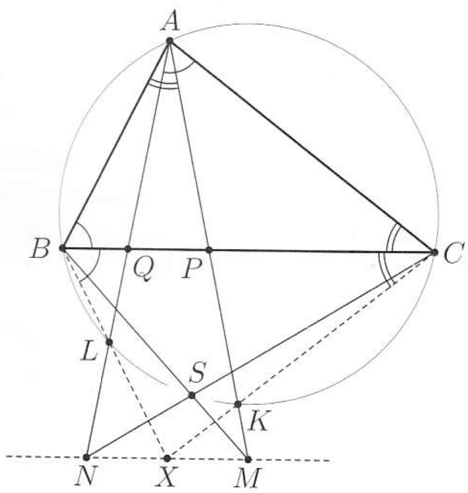

Figure 2

**Solution 2.** As in the previous solution, denote by $S$ the intersection point of the lines $BM$ and $NC$. Let moreover the circumcircle of the triangle $ABC$ intersect the lines $AP$ and $AQ$ again at $K$ and $L$, respectively (see Figure 2).

Note that $\angle LBC = \angle LAC = \angle CBA$ and similarly $\angle KCB = \angle KAB = \angle BCA$. It implies that the lines $BL$ and $CK$ meet at a point $X$, being symmetric to the point $A$ with respect to the line $BC$. Since $AP = PM$ and $AQ = QN$, it follows that $X$ lies on the line $MN$. Therefore, using PASCAL's theorem for the hexagon $ALBSCK$, we infer that $S$ lies on the circumcircle of the triangle $ABC$, which finishes the proof.

**Comment.** Both solutions can be modified to obtain a more general result, with the equalities

$$AP = PM \quad \text{and} \quad AQ = QN$$

replaced by

$$\frac{AP}{PM} = \frac{QN}{AQ}\,.$$