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A Generalization of an IMTS Problem

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In the *Mathematics & Informatics Quarterly*, Sept 1996, pp. 180-182 the following problem from the International Mathematical Talent Search (IMTS) 19 was given and solved in several different ways:

"In Figure 1 determine the area of the shaded octagon as a fraction of the area of the square, where the boundaries of the octagon are lines drawn from the vertices to the midpoints of opposite sides."

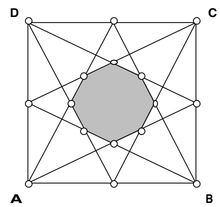

Figure 1

The last solution showed that the result (the area of the octagon is 1/6 of the area of the square) was more generally true for a parallelogram. This made me wonder whether the result could be further generalized by dividing the sides of the parallelogram into ratios other than two. A quick investigation with the dynamic geometry software *Sketchpad* showed that if the sides were divided into thirds as shown in Figure 2, then the area of the octagon remains in a constant ratio (apparently 1/3) to that of the parallelogram no matter how the size or shape of the parallelogram is changed.

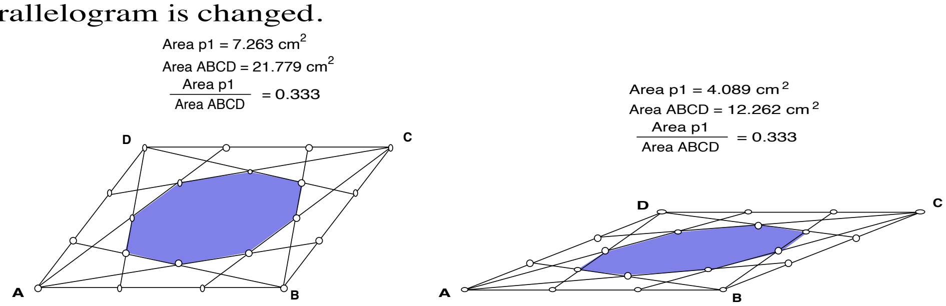

Figure 2

Furthermore, if the sides are divided into fourths as shown in Figure 3, then the area of the octagon also remains in a constant ratio (apparently 9/20) to that of the parallelogram no matter how the size or shape of the parallelogram is changed. This clearly implies the interesting, general conjecture that the area of the octagon is always in a constant ratio to that of the parallelogram, irrespective of the ratio into which the sides are divided!

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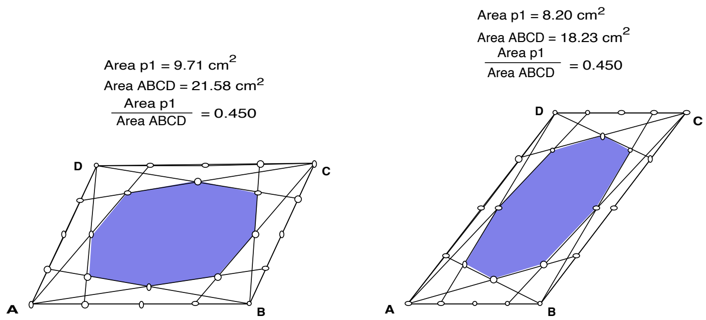

Figure 3

How can we prove this general conjecture? Is there a general formula for the ratio of the area of the octagon to that of the parallelogram?

This is perhaps a good example where the standard "problem solving" technique of just looking at the specific numerical ratios (eg. 1/6; 1/3; 9/20; etc.) to find a general formula is hardly likely to be successful. However, by first constructing separate proofs for the two examples shown above, I was able to develop the following general proof.

Lemma 1

Consider a triangle ABC with two sides AB and AC divided by points P and Q respectively, so that $AP = \frac{1}{p}AB$ and $AQ = \frac{1}{p}AC$ (see Figure 4). If O is the intersection of the two cevians BQ and CP, and a third cevian is drawn from A through O to intersect BC in Y, then $OY = \left(\frac{p-1}{p+1}\right)AY$.

Proof

Draw PQ and label its intersection with AY as X. Since P and Q divide AB and AC in the same ratio, it follows that PQ // BC. Therefore, $AX = \frac{1}{p}AY$ and $XY = \left(\frac{p-1}{p}\right)AY$.

But triangle PQO is similar to triangle CBO with an enlargement factor of p since $PQ = \frac{1}{p}BC$. Therefore:

$OX = \frac{1}{p}OY$ and $OY = \left(\frac{p}{p+1}\right)XY = \left(\frac{p}{p+1}\right)\left(\frac{p-1}{p}\right)AY = \left(\frac{p-1}{p+1}\right)AY$.

(*Note:* From Ceva's theorem it also follows that Y is the midpoint of BC (and therefore X is also the midpoint of PQ)).

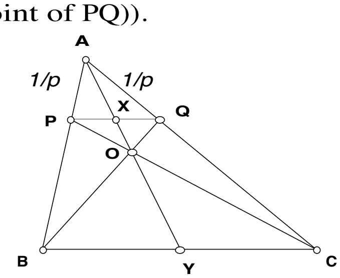

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Figure 4

Generalization

If the sides of any parallelogram are divided into p equal parts ($p > 1$), and an octagon is formed as shown in Figure 5, then the area of the octagon is equal to $\left(\frac{p-1}{p}\right)\left(\frac{p-1}{p+1}\right)$ that of the area of the parallelogram.

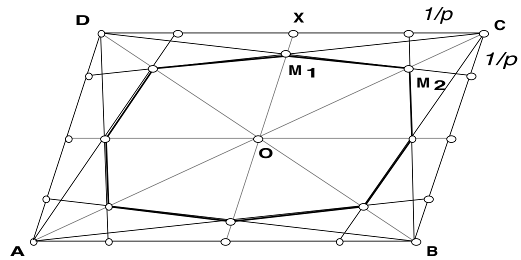

Figure 5

Proof

Connect the opposite vertices of the parallelogram, as well as the midpoints of the opposite sides as shown. Consider triangle XOC and observe that:
$M_1X = \left(\frac{1}{2p}\right)BC = \left(\frac{1}{2p}\right)2OX = \frac{1}{p}OX$. Therefore, $OM_1 = \left(\frac{p-1}{p}\right)OX$.

From Lemma 1, it also follows (in triangle BCD) that $OM_2 = \left(\frac{p-1}{p+1}\right)OC$. Therefore, the area of triangle OM_1M_2 is $\left(\frac{p-1}{p}\right)\left(\frac{p-1}{p+1}\right)$ that of the area of triangle OXC.

Similarly, one can show that the area of each triangular section of the octagon M is $\left(\frac{p-1}{p}\right)\left(\frac{p-1}{p+1}\right)$ that of the area of the corresponding triangular section of the parallelogram, and therefore completes the proof.

Note

Sketchpad is available from **Dynamic Learning**, 8 Cameron Rd, Pinetown 3610. Tel: 031-7083709/2044252; 031-7029941 (Pearl); e-mail: dynamiclearn@mweb.co.za; website: http://mzone.mweb.co.za/residents/profmd/homepage.html

Sketchpad is now completely free. More info at
URL: http://dynamicmathematicslearning.com/free-download-sketchpad.html

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The solutions to the first Sketchpad Competition can now be downloaded from http://mzone.mweb.co.za/residents/profmd/homepage3.html.

The above link is unfortunately no longer active.