An Interesting Collinearity

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INTRODUCTION

With reference to Figure 1, three rays are drawn from a point J to intersect two rays drawn from a point O. The points of intersection of the rays are shown (A, B, C, D, E, F). Quadrilaterals ECFB and ADFC are formed in this process. If we draw in the diagonals of these two quadrilaterals then points I (the point of intersection of the diagonals of ECFB), H (the point of intersection of the diagonals of ADFC) and O appear to be collinear.

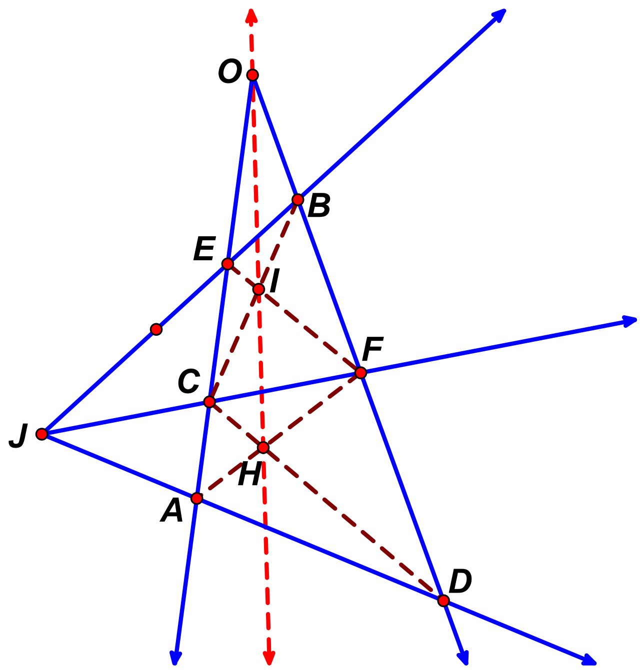

FIGURE 1: An interesting collinearity.

The above interesting observation/conjecture was discovered serendipitously in 2018 by the second author while developing elementary geometry materials about lines for primary school learners. The first author was then approached to confirm the result and to assist with proving it.

A dynamic geometry version of the construction illustrated in Figure 1 can be found at:

http://dynamicmathematicslearning.com/interesting-collinearity.html

Before continuing, readers are invited to use this link to explore and ascertain for themselves the validity of the conjecture.

TWO FAMOUS THEOREMS

In order to prove the result, two famous geometry theorems are needed – the Theorem of Desargues and the Theorem of Pappus. Each of these two important theorems is described below without proof [3].

THE THEOREM OF DESARGUES

This theorem (illustrated in Figure 2) states that two triangles are point perspective if and only if they are line perspective. The figure shows triangles ABC and $A'B'C'$ in perspective from O, with the respective intersections X, Y and Z of the extensions of their corresponding sides being collinear.

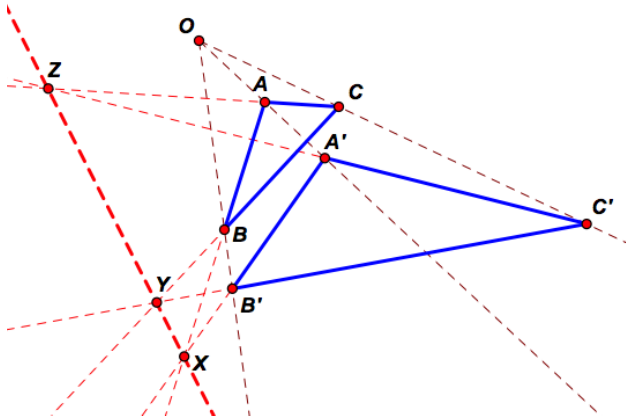

FIGURE 2: Theorem of Desargues.

This remarkable theorem is named after Girard Desargues, a French architect and engineer in the 17th century. A friend and pupil of his, Abraham Bosse, in a practical book on the use of perspective, first published the theorem in 1648. For interest, one way of proving Desargues' theorem in a surprisingly easy manner is first to prove the 3D version of it as described in De Villiers and Garner (2008), and then simply to consider the 2D version of the theorem as the projection of it onto the plane.

THE (HEXAGON) THEOREM OF PAPPUS

This theorem states that if the vertices of a hexagon ABCDEF alternately lie on two lines, then the intersections of the three pairs of opposite sides (AB and DE; BC and EF; CD and FA) are collinear (Figure 3). This equally remarkable theorem is named after Pappus of Alexandria who lived from about 290 AD to 350 AD, and was one of the last great Greek mathematicians of antiquity.

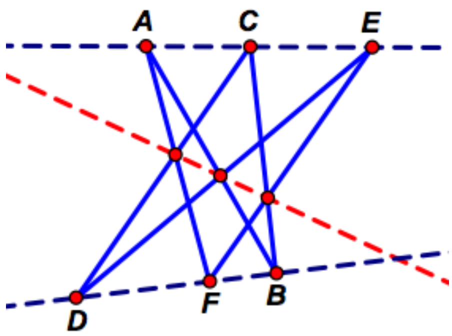

FIGURE 3: Theorem of Pappus.

[3] For more information about these two theorems go to: https://en.wikipedia.org/wiki/Desargues%27s_theorem and https://en.wikipedia.org/wiki/Pappus_of_Alexandria

In modern terms, the Theorem of Desargues and the Theorem of Pappus both belong to projective geometry, which was only established formally in the 19th century by Poncelet, Monge, Steiner, and others. In projective geometry, unlike Euclidean geometry, angle size, length, congruency, similarity, parallelness, etc. are no longer invariant. The only invariant properties (among a few others) that are investigated are concurrency of lines and collinearity of points.

PROVING THE CONJECTURE

We are now in a position to prove the original conjecture. To prove that the three points H, I and O are collinear we can use Desargues' theorem as follows (Figure 4). Note that triangles EFA and BCD are in perspective from J as a result of the construction. Thus, I, H and O, the respective intersections of the corresponding sides of these two triangles, are collinear.

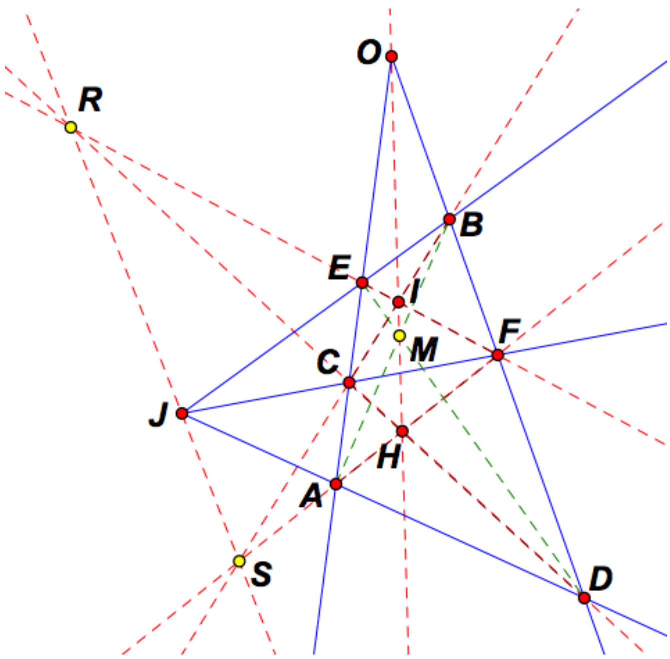

FIGURE 4: Proving the conjecture.

Note also that since the vertices of the hexagon ABCDEF alternately lie on two lines, it also follows directly from the theorem of Pappus that the intersections H, M and I of its opposite sides are collinear. Hence, the four points H, M, I and O are collinear. Lastly, since triangles EIB and AHD are point perspective from O, it once again follows from the theorem of Desargues that the three intersections R, J and S of their corresponding sides are also collinear.

CONCLUDING REMARK

The original problem is a pleasing, straightforward application of these two famous theorems, and would present a good practice challenge for learners preparing for the Third Round of the SA Mathematics Olympiad. It could also be used as enrichment to the high school geometry curriculum in a Mathematics Club. Due to the richness of the diagram, readers may even find additional interesting geometry properties.

REFERENCES

De Villiers, M. & Garner, M. (2008). Problem-solving and proving via generalization. *Learning & Teaching Mathematics*, *5*, pp. 19-25.