An Interesting Locus Result

Duncan Samson[1] & Moshe Stupel[2]

[1]Victoria Girls' High School, Makhanda (Grahamstown)

[2]Givat Washington Academic College, Israel

dsamson@vghs.co.za   stupel@bezeqint.net

INTRODUCTION

Senior high school pupils will be familiar with the "midpoint theorem" – the Euclidean geometry theorem that states that the line joining the midpoints of two sides of a triangle is parallel to the third side (and equal to half the length of the third side). This well-known theorem is illustrated in Figure 1.

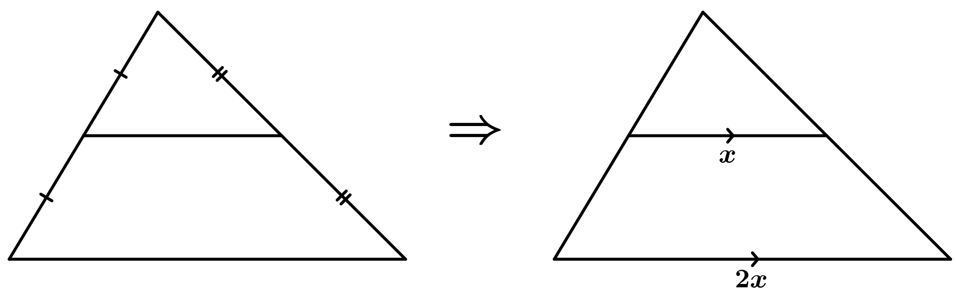

FIGURE 1: The midpoint theorem.

A somewhat less intuitive, but related scenario, is the following. Consider a straight line f and a fixed point A not on f. Create a series of straight-line segments by joining point A to a variety of different points on f as illustrated in Figure 2.

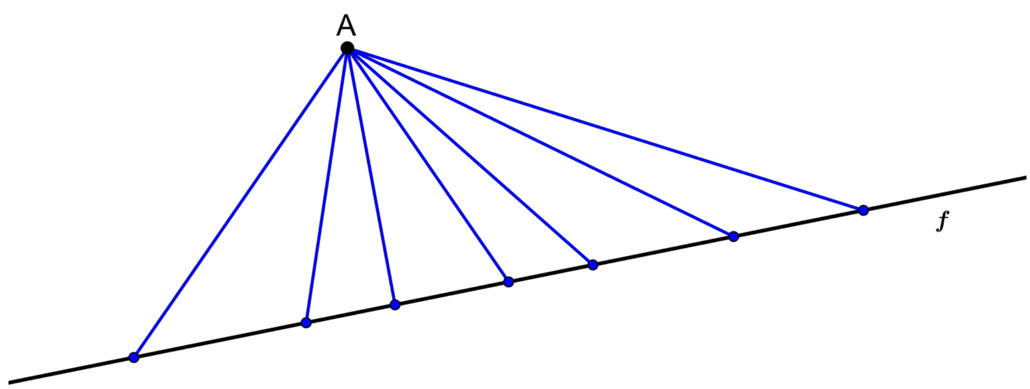

FIGURE 2: Straight-line segments from A to f.

Next construct the midpoint of each of these line segments. You may be surprised to find that not only are all of these midpoints collinear, but the straight line passing through them (g) is parallel to the original straight line f (Figure 3). Linking this scenario back to the midpoint theorem sheds light, from a geometric perspective, on why the midpoints are collinear as well as why the line passing through them is parallel to f.

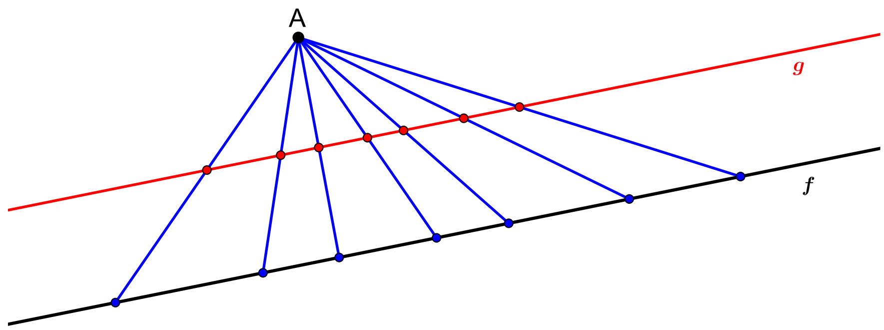

FIGURE 3: Collinear midpoints.

How could we go about determining the equation of g, the line passing through the midpoints? Let us take the equation of f to be $y_0 = mx_0 + c$, and the coordinates of A to be $(a; b)$. What we now need to determine is the equation representing the locus of all points $(x; y)$ lying on the midpoints of the line segments drawn from A to f.

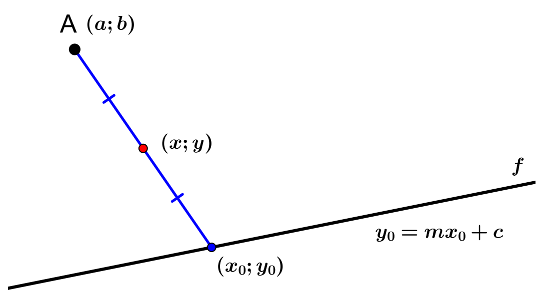

FIGURE 4: Determining the locus of points $(x; y)$ lying on the midpoints of the line segments.

Since $(x; y)$ is the midpoint of the line segment, we have:

$$x = \frac{a + x_0}{2} \quad \rightarrow \quad x_0 = 2x - a$$

$$y = \frac{b + y_0}{2} \quad \rightarrow \quad y_0 = 2y - b$$

We can now substitute the above expressions for x_0 and y_0 as follows:

$$y_0 = mx_0 + c$$

$$\therefore \quad 2y - b = m(2x - a) + c$$

$$\therefore \quad 2y = 2mx - ma + c + b$$

$$\therefore \quad y = mx + \frac{b + c - ma}{2}$$

This is the equation of the locus of all points $(x; y)$ lying on the midpoints of the line segments drawn from A to f. Note that the gradient of this line is also m, as it was for the original line f. This then confirms, from an algebraic perspective, not only that the midpoints lie on a straight line, but that this line is parallel to the original straight line.

EXTENDING THE IDEA

If this works for straight lines, would it also work for other functions? Let us consider a simple parabola such as $f(x) = \frac{1}{4}x^2$. As before, plot a fixed point A not on the parabola and create a series of straight-line segments from A to a variety of different points on f. Now construct the midpoint of each of these straight-line segments. Rather pleasingly, these midpoints also line up in the form of a parabola, irrespective of the position of the fixed point A (Figure 5).

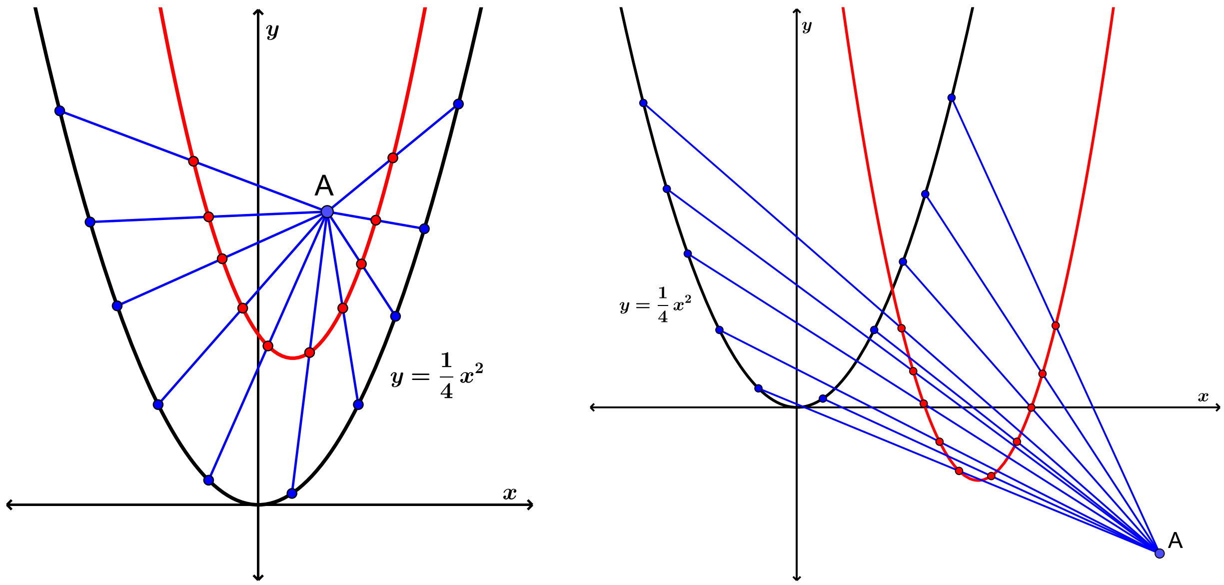

FIGURE 5: Joining the midpoints.

Let us now try to confirm algebraically that the midpoints do indeed line up on a parabolic curve. Additionally, let us try to establish what the relationship is between the original parabola and that represented by the locus of points lying on the midpoints of the line segments.

Let us take the equation of the original parabola to be $y_0 = kx_0{}^2$, and the coordinates of A to be $(a; b)$. What we now need to determine is the equation representing the locus of all points $(x; y)$ lying on the midpoints of the line segments drawn from A to $y_0 = kx_0{}^2$.

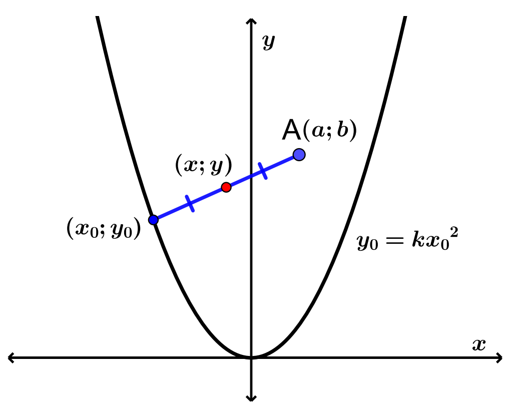

FIGURE 6: Determining the locus of points $(x; y)$ lying on the midpoints of the line segments.

Since $(x; y)$ is the midpoint of the line segment, we have:

$$x = \frac{a + x_0}{2} \quad \rightarrow \quad x_0 = 2x - a$$

$$y = \frac{b + y_0}{2} \quad \rightarrow \quad y_0 = 2y - b$$

We can now substitute the above expressions for x_0 and y_0 as follows:

$$y_0 = k{x_0}^2$$

$$\therefore \quad 2y - b = k(2x - a)^2$$

$$\therefore \quad 2y = k(2x - a)^2 + b$$

$$\therefore \quad 2y = k\left(2\left(x - \frac{a}{2}\right)\right)^2 + b$$

$$\therefore \quad y = 2k\left(x - \frac{a}{2}\right)^2 + \frac{b}{2}$$

This is the equation of the locus of all points $(x; y)$ lying on the midpoints of the line segments drawn from A to the parabola. The equation confirms that the locus of points is indeed parabolic. Note also that the axis of symmetry of the new parabola is parallel to the axis of symmetry of the original parabola. Additionally, note that the "stretch factor" of the new parabola is twice that of the original ($2k$ versus k in the original).

While we have only considered simple parabolas of the form $y = kx^2$, i.e. that have a turning point at the origin, the result would still hold true for parabolas containing a vertical and/or horizontal shift.

CONCLUDING COMMENTS

While the midpoint theorem is familiar to senior high school pupils, reimagining this idea from a slightly different perspective (i.e. exploring the locus of midpoints of a series of straight-line segments drawn from a fixed point to an arbitrary straight line) leads to an interesting, and perhaps less intuitive, result – namely that the locus of midpoints is a straight line parallel to the original straight line. We then took this basic idea and extended it to the case of a simple parabola with turning point at the origin and showed that in this case the locus of midpoints also forms a parabola with some interesting relationships to the original parabola. Similar locus observations occur in the case of the circle, ellipse and hyperbola, as well as other curves. Furthermore, while we have only considered the midpoints of the line segments, any fixed division of the straight-line segments would work.

One of the great joys of mathematics is how different concepts link, and how looking at a given scenario from different perspectives can lead to different insights and deeper understanding. With reference to Figure 5, we could reimagine the process as a dilation (reduction) of the parabola by a factor ½ from point A. From this transformation perspective it becomes clear why the process would work for any curve[1], as well as any fixed division of the straight-line segments.

---

[1] See for example the following webpage by Michael de Villiers: http://dynamicmathematicslearning.com/mystery-transform.html