# Intersecting Circles Investigation

TechSpace

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The reader is first invited to explore a dynamic geometry sketch of the investigation at: http://dynamicmathematicslearning.com/intersecting-circles.html where the stage is set for conjecturing and proving interesting properties of intersecting circles.

For those readers who would like to create their own sketches in GeoGebra, the following instructions may be helpful.

and intersecting the other circle at (0, 1).

| Construction Steps for GeoGebra | |
|---|---|
| Step 1: | Construct a unit circle with its centre at the origin and with a radius of 1. |
| Step 2: | Construct another circle with its centre OB on the x-axis ~~and with radius 1 unit.~~ |
| Step 3: | Construct (and ~~name) points~~ B and B' where the circle with centre OB intersects the x-axis and measure the x-coordinates of these points. |
| Step 4: | Construct a perpendicular to the x-axis at OB, and then label its intersection with the circle in the 1st quadrant as H. |
| Step 5: | Measure the y-coordinate of the point H. |
| Step 6: | Trace the path of point H when OB is dragged. |

Figure 1 shows four intersecting circles that share a common vertical chord that is the diameter of a unit circle whose center is at the origin. Their intersections on the negative $x$-axis are separated by equal intervals of $\frac{1}{4}$. The four arcs to the left of 0 may remind you of longitude lines on a two-dimensional rendering of a globe, which was the inspiration for this investigation. These few examples satisfy the conjecture that the $x$-coordinates of the endpoints of the diameters lying on the $x$-axis of each circle are negative reciprocals of each other. A proof for the general situation follows.

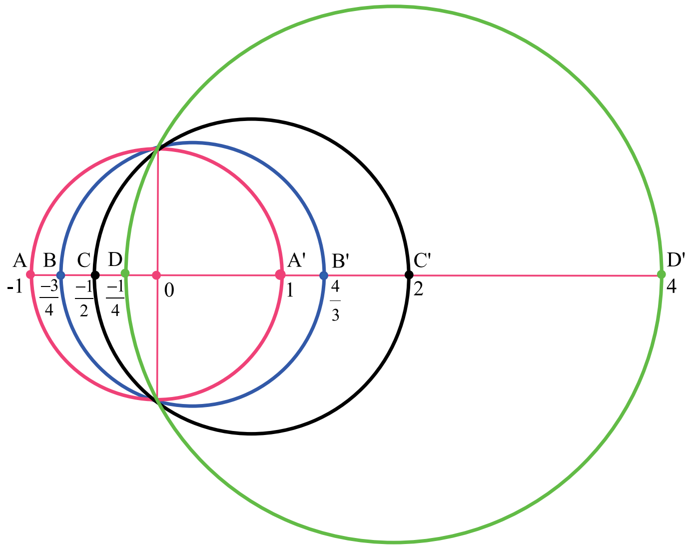


Figure 1. Four intersecting circles, with $x$-coordinates on the $x$-axis noted

**General Situation**

Figure 2 shows unit circle A with center at (0,0). Circle B with center D($x$,0) is drawn to pass through point C(0,1) and the point B with coordinates ($-p$,0), with $p$ a non-zero real number. Prove that the $x$-coordinates of the endpoints of the diameter on the $x$-axis of circle B are negative reciprocals of each other.

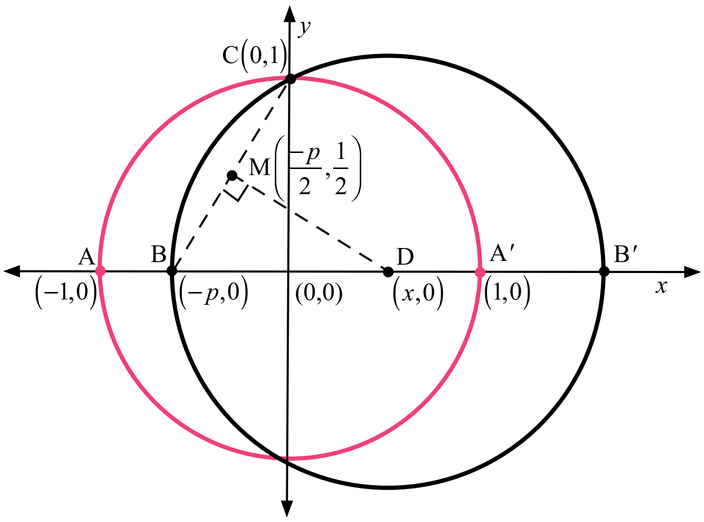


Figure 2. General proof

**Proof.** Point M is the midpoint of $\overline{CB}$ and has coordinates M $\left(\frac{-p}{2}, \frac{1}{2}\right)$.

The slope of $\overline{CB}$ is $\frac{1}{p}$, so the slope of $\overline{MD}$ is $-p$. Then $\dfrac{\frac{1}{2}-0}{\frac{-p}{2}-x} = -p$, so $x = \dfrac{1-p^2}{2p}$.

The radius of circle B is $x+p$ so the radius of circle B is $\dfrac{1+p^2}{2p}$.

Then $x + \text{radius} = \dfrac{1-p^2}{2p} + \dfrac{1+p^2}{2p} = \dfrac{1}{p}$, so B′ has coordinates $\left(\dfrac{1}{p}, 0\right)$ as required.

### The Converse

We now prove the converse:

> Given a circle with center on the $x$-axis and $x$-intercepts that are negative reciprocals of each other, the circle intersects (0,1) and (0, −1).

**Proof.** The center of the circle is at $\left(\dfrac{-p+\frac{1}{p}}{2}, 0\right)$ or $\left(\dfrac{1-p^2}{2p}, 0\right)$. The radius of the circle is $\dfrac{\frac{1}{p}-(-p)}{2} = \dfrac{1+p^2}{2p}$. Thus, the equation of the circle is $\left(x-\dfrac{1-p^2}{2p}\right)^2 + y^2 = \left(\dfrac{1+p^2}{2p}\right)^2$. To determine the $y$-intercepts, replace $x$ with 0 and solve for $y$. Hence, $\left(-\dfrac{1-p^2}{2p}\right)^2 + y^2 = \left(\dfrac{1+p^2}{2p}\right)^2$, so $y^2 = \left(\dfrac{1+p^2}{2p}\right)^2 - \left(-\dfrac{1-p^2}{2p}\right)^2 = \dfrac{4p^2}{4p^2} = 1$, so $y = \pm 1$, proving that the circle passes through (0,1) and (0, −1).

### Connection with Inversive Geometry

A closely related geometric idea to finding the reciprocal of a number is that of "inverting" a point. In the plane, the inverse of a point P (lying inside, on or outside a reference circle with center O and radius $r$) is defined as a point P′, lying on the ray from O through P such that $OP \cdot OP' = r^2$ (see Figure 3).

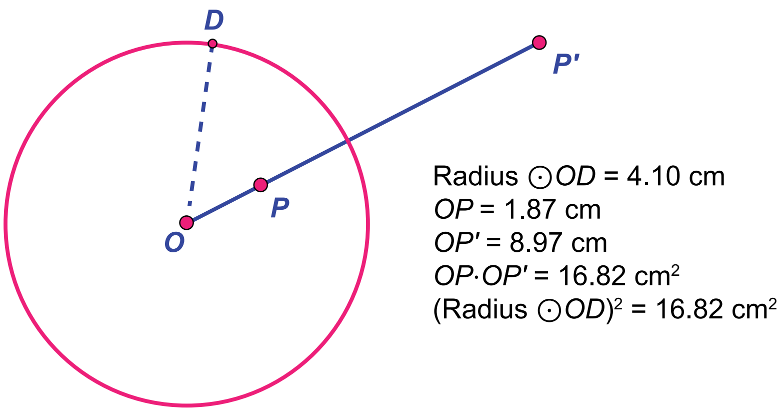


Figure 3. Definition of Inversion in Geometry

Now, with reference to Figure 2, if we reflect point B in the $y$-axis and label the image B″, then it follows from the already proven result at the beginning that $OB' \cdot OB'' = 1 = r^2$, where $r = 1$ is the radius of the unit circle. Hence, by definition B″ is simply the inversion of B′ in the unit circle, and vice versa.

The interested reader can read more about Inversive Geometry at:
https://en.wikipedia.org/wiki/Inversive_geometry

### High Points

Consider Figure 4, which is a screen capture of the dynamic sketch at the URL given at the start. If the center of the circle, $O_B$, is $(x,0)$, the radius from $O_B$ to $(0,1)$ is the hypotenuse of a right triangle, so the height, $y$, of the circle above the $x$-axis is given by $y = \sqrt{x^2+1}$, the upper branch of a hyperbola. The lower branch, with equation $y = -\sqrt{x^2+1}$, passes through the lowest point of the circle. The asymptotes are

$$y = \pm x.$$

### Pythagoras and Triples

We next discover a proof of the Pythagorean theorem and relate $p$ to Pythagorean triples. Suppose $p$ is a non-zero rational number $\frac{n}{m}$, with $m > n$. The center of the circle is $\left(\frac{m^2-n^2}{2mn}, 0\right)$ and the radius is $\frac{m^2+n^2}{2mn}$, so the highest point on the circle is $\left(\frac{m^2-n^2}{2mn}, \frac{m^2+n^2}{2mn}\right)$. Define positive integers $a = m^2 - n^2$, $b = 2mn$ and $c = m^2 + n^2$. Then the highest point on a circle has coordinates $\left(\frac{a}{b}, \frac{c}{b}\right)$. Figure 5 shows that $\frac{a}{b}$, $\frac{b}{b}$ and $\frac{c}{b}$ are the three sides of a right triangle, the same right triangle we used to establish the equation of the curve passing through the highest points of the circles.

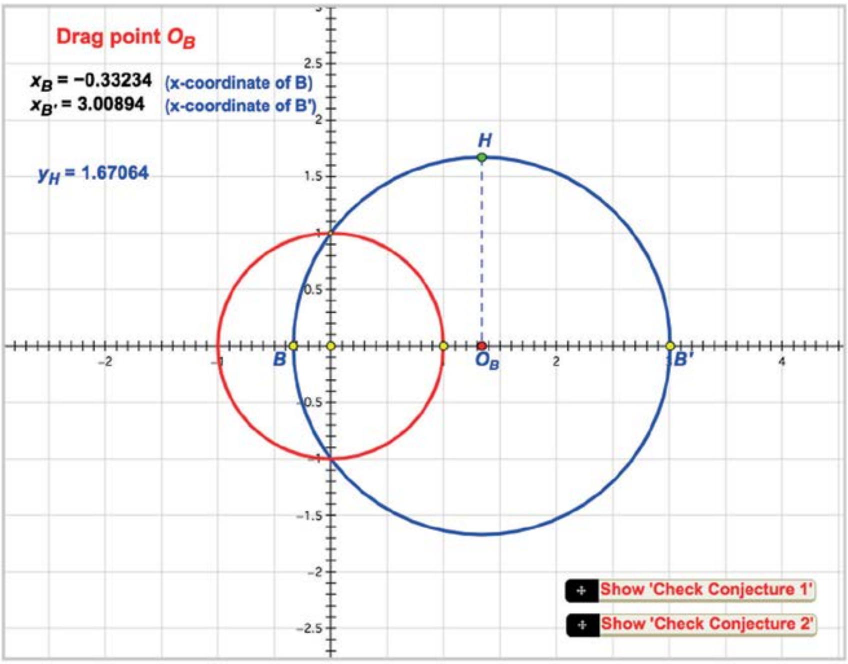


Figure 4. Dynamic Sketch

Thus, $2m^2 = c + a$, so $m = \sqrt{\frac{c+a}{2}}$ and $2n^2 = c - a$ so $n = \sqrt{\frac{c-a}{2}}$. Then $\frac{n}{m} = \sqrt{\frac{c-a}{c+a}}$. Furthermore, the difference between the radius and the $x$-coordinate of the center, $\frac{c-a}{b}$, is also equal to $\frac{n}{m}$, hence $\sqrt{\frac{c-a}{c+a}} = \frac{c-a}{b}$. The last equation becomes $\frac{c-a}{c+a} = \frac{(c-a)^2}{b^2}$, so $\frac{1}{c+a} = \frac{c-a}{b^2}$ and thus $a^2 + b^2 = c^2$, or equivalently $\left(\frac{a}{b}\right)^2 + \left(\frac{b}{b}\right)^2 = \left(\frac{c}{b}\right)^2$, which relates the three sides of the right triangle in Figure 3, thereby proving the Pythagorean theorem, since the argument holds for all positive real values of $a$, $b$ and $c$.

Looking at $\left(\frac{m^2-n^2}{2mn}, \frac{m^2+n^2}{2mn}\right)$ we can clearly see what Euclid taught us so many years ago, that for positive integers $m$ and $n$, $m > n$, with $a = m^2 - n^2$, $b = 2mn$, and $c = m^2 + n^2$, $a$, $b$ and $c$ form a Pythagorean triple.

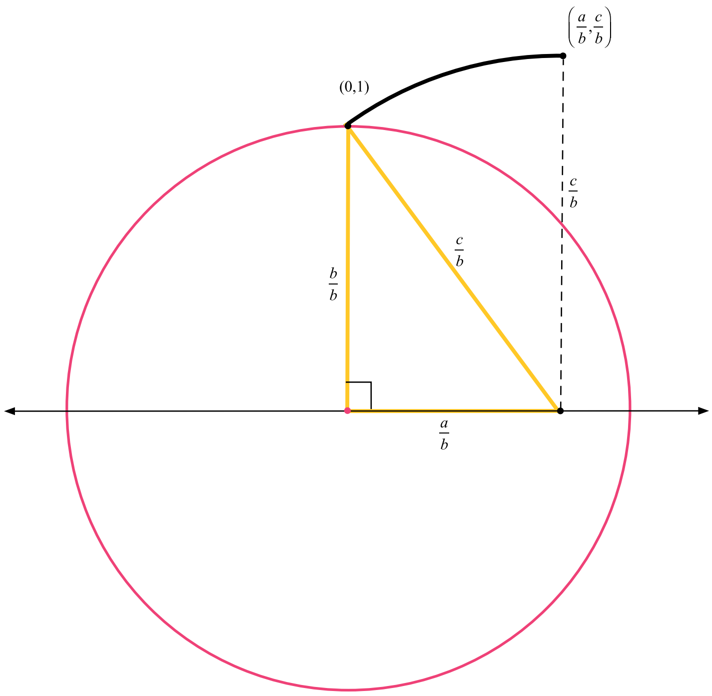


Figure 5. Stumbling upon a proof of Pythagoras theorem

If we consider only positive integer values of $n$ and $m$, then for each of our circles we can use the known values of $n$ and $m$ to determine the coordinates of the highest point on the circle, which in turn can tell us a corresponding Pythagorean triple.

For example, the circle passing through $\left(\frac{-3}{4}, 0\right)$ will have its highest point at $\left(\frac{7}{24}, \frac{25}{24}\right)$ revealing the Pythagorean triple 7–24–25.

Furthermore, we can start with a Pythagorean triple and determine the circle. Suppose we select the Pythagorean triple 9–40–41. Since $n = \sqrt{\frac{c-a}{2}}$ and $m = \sqrt{\frac{c+a}{2}}$, $n = 4$ and $m = 5$. This means that a circle with center on the $x$-axis through $\left(\frac{-4}{5}, 0\right)$ will have the highest point of the circle at $\left(\frac{9}{40}, \frac{41}{40}\right)$. The other circles in Figure 1 have high points at $\left(\frac{15}{8}, \frac{17}{8}\right)$, generating the 15–8–17 triangle and $\left(\frac{12}{16}, \frac{20}{16}\right) = \left(\frac{3}{4}, \frac{5}{4}\right)$ generating the 12–16–20 triangle, a multiple of the 3-4-5 primitive Pythagorean triple, which comes from $n = 1$ and $m = 2$.

Figure 6 shows the circles from Figure 1 and the coordinates of their highest points.

Our investigation of lines of longitude has taken us from negative reciprocals to inversive geometry, to a hyperbola whose branches contain the highest and lowest points of a series of circles, to a proof of the Pythagorean theorem and finally a connection to Pythagorean triples.

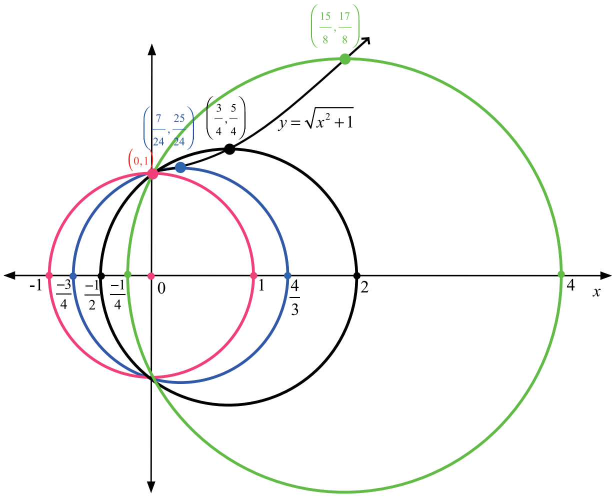


Figure 6. Graph of $y = \sqrt{x^2 + 1}$ with 4 circles

The pedagogic benefits of such an exploration are manifold. The dragging facility of dynamic geometry software is valuable because it quickly generates numerous examples that comply with the constraints of a particular construction. Not only can dragging produce numerical data, but it also provides visual stimuli for conjecturing.

While dynamic geometry provides valuable experimental confirmation, a deductive proof is needed to explain and understand why one's observations are true, which refers to the explanatory function of proof (De Villiers, 1999). More-over, since the software cannot check all possible cases, one ultimately needs a deductive proof to verify that the conjecture is generally true.

In this case, after observing the result about the product of the x-coordinates as well as the path traced out by H, we needed a proof to generally verify, as well as to explain our observations. Further reflection on the proof of the results led us to Pythagorean triples and a proof of the theorem of Pythagoras, which also nicely illustrates the discovery function of proof (De Villiers, 1999).


**MICHAEL DE VILLIERS** has worked as researcher, mathematics and science teacher at institutions across the world. From 1983-1990, he was at the University of Stellenbosch, and from 1991-2016 he was part of the University of KwaZulu-Natal. After retirement in 2016, he was appointed Honorary Professor in Mathematics Education at the University of Stellenbosch. He was editor of Pythagoras, the research journal of the Association of Mathematics Education of South Africa, and is currently chair of the Senior South African Mathematics Olympiad problems committee. His main research interests are Geometry, Proof, Applications and Modeling, Problem Solving, and the History of Mathematics. His home page is http://dynamicmathematics-learning.com/homepage4.html. He maintains a web page for dynamic geometry sketches at http://dynamic-mathematicslearning.com/JavaGSPLinks.htm. He may be contacted at profmd1@mweb.co.za.



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# NUMBER FLEXIBILITY

Breaking down numbers to your advantage for problem solving is called number flexibility, which is a prerequisite for doing good math. Let's look at this with an example. In how many ways can you compute 18 × 5?

a. Repeated addition: 18 + 18 + 18 + 18 + 18 =90

b. Doubling to shorten the above: 18 × 2 × 2 + 18 → 36 → 72 → 90 (Double 18 twice and then add 18)

c. Using distributive property:

i. (20 – 2) × 5 = 100 – 10 = 90

ii. (12 + 6) × 5 = 60 + 30 = 90

iii. (9 + 9) × 5 = 45 + 45 = 90

d. The last method is just halving and doubling: (18 ÷ 2) × 5 × 2 = 45 × 2 = 90

e. Or it can be doubling and halving: 18 × (5 × 2) ÷ 2 = 180 ÷ 2 = 90

**Why are d. and e. equivalent?**

Halving is the same as multiplication by ½ and multiplication is both commutative and associative.

Let's look at some more examples of doubling and halving. Which number is doubled and which is halved? How does this make the calculation simpler?

a. 12 × 50 = (12 ÷ 2) × (50 × 2) = 6 × 100

b. 16 × 25 = 8 × 50 = 4 × 100

When do we use the distributive property: 24 × 11 = 24 × (10 + 1) = 240 + 24

Now, in how many ways can you compute 12 × 15? Which do you find the easiest?

*Contributed by Vikas Sharma*